Explicit formulas for generators of triangular norms

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Definitions

Triangular norm (t-norm)

An operation $T\colon [0,1]\times[0,1]\to[0,1]$ such that for all $x,y,z\in[0,1]$:

(T1) $T(x,y)=T(y,x)$ (commutativity)

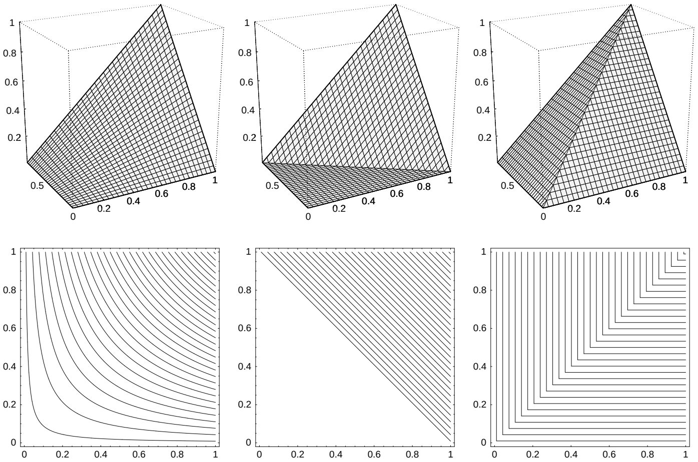
(T2) $T(x,T(y,z))=T(T(x,y),z)$ (associativity)

(T3) $x\le y\Rightarrow T(x,z)\le T(y,z)$ (monotonicity)

(T4) $T(x,1)=x$ (neutral element)

- BL (Basic logic) and MTL (Monoidal t-norm based logic)
  - logical conjunctions in standard semantics
- probabilistic metric spaces
  - triangular inequality

Examples of t-norms



- $T_{\mathbf{P}}(x, y) = x \cdot y$
- $T_{\mathbf{L}}(x, y) = \max\{x + y - 1, 0\}$
- $T_{\mathbf{M}}(x, y) = \min\{x, y\}$

Definitions

Natural power of x

- $T \ldots$ t-norm, $n \in \mathbb{N}$, $x \in [0,1]$

$$\begin{aligned} x_T^{(0)} &= 1 \\ x_T^{(1)} &= x \\ x_T^{(n)} &= T(\underbrace{x, x, \ldots, x}_{n\text{-times}}) \text{ if } x \geq 2 \end{aligned}$$

Archimedean t-norm

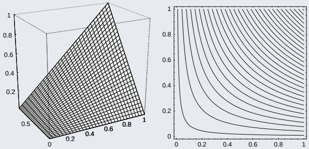
- for every $x, y \in \,]0,1[$, $x < y$, there exists $n \in \mathbb{N}$ such that $y_T^{(n)} < x$

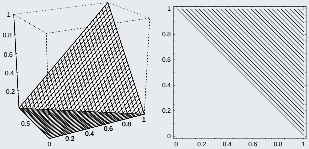
Continuous Archimedean t-norm

- $T(x,x) < x$ for all $x \in \,]0,1[$

Continuous Archimedean t-norms

- *strict* ... continuous and strictly increasing on $]0,1]^2$
- *nilpotent* ... for every $x \in]0,1[$ there exists $n \in \mathbb{N}$ such that $x_T^{(n)} = 0$
- continuous Archimedean t-norm ... either strict or nilpotent





- $T_{\mathbf{P}}(x,y) = x \cdot y$ (strict)
- $T_{\mathbf{L}}(x,y) = \max\{x+y-1, 0\}$ (nilpotent)

Definitions

φ-transform

- T ... t-norm, $\varphi\colon [0,1] \to [0,1]$ ... increasing bijection
- φ-transform of T:

$$T_\varphi\colon [0,1]^2 \to [0,1] : (x,y) \mapsto \varphi^{-1}\Big(T\big(\varphi(x),\varphi(y)\big)\Big)$$

- strict t-norm ... φ-transform of $T_{\mathbf{P}}$
- nilpotent t-norm ... φ-transform of $T_{\mathbf{L}}$

Multiplicative generator of a t-norm T

- a strictly increasing function $\theta\colon [0,1] \to [0,1]$ such that $\theta(1) = 1$ and

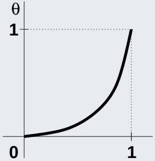
$$T(x,y) = \theta^{(-1)}\left(\theta(x) \cdot \theta(y)\right)$$

- $\theta^{(-1)}\colon [0,\infty] \to [0,1]$ … pseudo-inverse of θ:

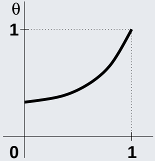
$$\theta^{(-1)}(y) = \begin{cases} 0 & \text{if} \quad y < \theta(0) \\ \theta^{-1}(y) & \text{if} \quad y \geq \theta(0) \end{cases}$$

Examples of multiplicative generators of:

strict t-norm:



nilpotent t-norm:



Definitions

Additive generator of a t-norm T

- a strictly decreasing function $t\colon [0,1] \to [0,\infty]$ such that $t(1) = 0$ and

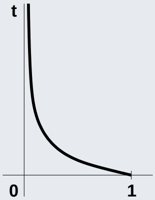
$$T(x,y) = t^{(-1)}\left(t(x) + t(y)\right)$$

- $t^{(-1)}\colon [0,\infty] \to [0,1] \ldots$ pseudo-inverse of t:

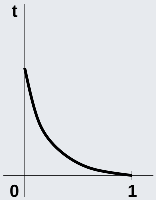
$$t^{(-1)}(y) = \begin{cases} 0 & \text{if } \; y > t(0) \\ t^{-1}(y) & \text{if } \; y \le t(0) \end{cases}$$

Examples of additive generators of:

strict t-norm:



nilpotent t-norm:



Properties of generators

Set of generators of a given t-norm

Let T be a continuous Archimedean t-norm.

- Let θ be its multiplicative generator;
  $T(x,y) = \theta^{(-1)}(\theta(x) \cdot \theta(y))$.
  - For every $q \in]0,\infty[$, θ^q is a multiplicative generator of T.
- Let t be its additive generator;
  $T(x,y) = t^{(-1)}(t(x) + t(y))$.
  - For every $p \in]0,\infty[$, $p \cdot t$ is a additive generator of T.

Conversion

Let T be a continuous Archimedean t-norm.

- If θ is a multiplicative generator of T
  - then $-\ln\theta$ is an additive generator of T.
- If t is a additive generator of T
  - then e^{-t} is a multiplicative generator of T.

Representation of continuous Archimedean t-norms

Theorem

Every continuous Archimedean t-norm can be represented by a multiplicative and an additive generator.

Proof

- given by Mostert and Shields in 1957
- constructive
- not intuitive
- does not allow to express the generator by a formula
- some alternatives:
  - Craigen and Páles in 1989
  - Pi-Calleja in 1954, published by Alsina in 1992
    - both methods allow to express the generator by an explicit formula
    - complicated

Two methods:

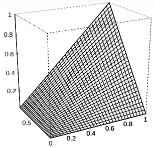
- Reconstruction of multiplicative generators of strict t-norms
- Reconstruction of additive generators of continuous Archimedean t-norms

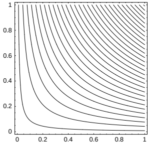
Definitions

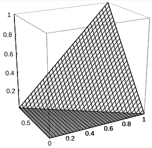
Support of t-norm T

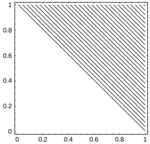
Supp T … closure of the set

$$\Big\{(x, y) \in [0, 1]^2 \mid T(x, y) > 0\Big\}$$









Notation for the first partial derivative

$$T'(x, y) = \left.\frac{\partial T(z, y)}{\partial z}\right|_{z=x} = \lim_{h \to 0} \frac{T(x+h, y) - T(x, y)}{h}$$

- derivatives of t-norm T will be considered only with respect to support Supp T

First derivative of inverse function

Lemma

Let f be a real function which is differentiable on an interval I. Let f possess an inverse function g. Each point $z \in I$ where g is differentiable and $f'(g(z)) \neq 0$ satisfies

$$g'(z) = \frac{1}{f'(g(z))} .$$

- $T(x,y) = \theta^{(-1)}(\theta(x) \cdot \theta(y)) = t^{(-1)}(t(x) + t(y))$
- If $(x,y) \in \operatorname{Supp} T$ then pseudoinverses coincide with inverses.

$$T'(x,y) = \frac{\theta(y) \cdot \theta'(x)}{\theta'(T(x,y))} = \frac{t'(x)}{t'(T(x,y))}$$

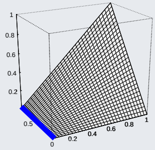
Reconstruction of multiplicative generators

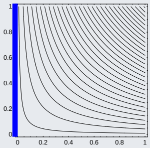
Motivation example

- substitute $x := 0$ in $T'(x,y) = \frac{\theta(y)\cdot\theta'(x)}{\theta'(T(x,y))}$:

$$T'(0,y) = \frac{\theta(y)\cdot\theta'(0)}{\theta'(0)} = \theta(y)$$

- We obtain directly the multiplicative generator.
- applicable if:
  - T is strict
  - $\theta'(0) \in]0,\infty[$ ... (not necessarily)





Reconstruction of multiplicative generators

Theorem

- T ... strict t-norm
- $f\colon\ ]0,1[\ \to [0,1] : y \mapsto T'(0,y) = \lim_{x \to 0_+} \frac{T(x,y)}{x}$
    - well defined on the whole $]0,1[$
- Then:
    - either $f(y) = 0$
    - or f is a bijection on $]0,1[$
    - or $f(y) = 1$

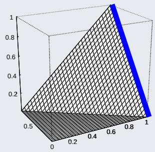
- Moreover, exteding f to the whole $[0,1]$ we obtain a multiplicative generator of T.

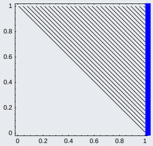
Motivation example

- substitute $x := 1$ in $T'(x,y) = \frac{t'(x)}{t'(T(x,y))}$:

$$T'(1,y) = \frac{t'(1)}{t'(y)} = \frac{b}{t'(y)}, \quad t'(y) = \frac{b}{T'(1,y)}$$

- We obtain the first derivative of the additive generator.
- $b \ldots$ any negative constant, WLOG -1
- applicable if:
  - $t'(1) \in]-\infty, 0[$





Reconstruction of additive generators

Residuum (residuated implication) of t-norm T

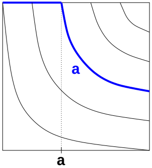
$I_T\colon\ [0,1]^2 \to [0,1]$

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\}$$

The case of continuous t-norms ... Divisibility

$$T(x, I_T(x,y)) = \min\{x,y\}$$

$$x\&(x \to y) = \min\{x,y\}$$



$x \mapsto I_T(x,a),\ a \in [0,1]$

Motivation example

- substitute $x := a \in]0,1]$ in $T'(x,y) = \frac{t'(x)}{t'(T(x,y))}$:

$$T'(a,y) = \frac{t'(a)}{t'(T(a,y))}$$

- use a new variable $z = T(a,y) \in [0,a] \Rightarrow y = I_T(a,z)$:

$$T'\left(a, I_T(a,z)\right) = T'(a,y) = \frac{t'(a)}{t'(T(a,y))} = \frac{t'(a)}{t'(z)} = \frac{b}{t'(z)}$$

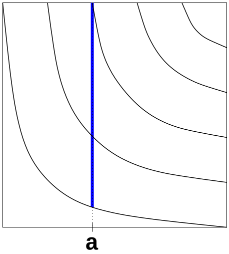
$$t'(z) = \frac{b}{T'(a,I_T(a,z))}$$

- b ... any negative constant, WLOG -1
- applicable if $t'(a) \in]-\infty, 0[$ and $z \in [0,a]$

Reconstruction of additive generators

Motivation example

$$t'(z) = \frac{b}{T'\left(a, I_T(a,z)\right)}\,, \quad z \in [0,a]$$



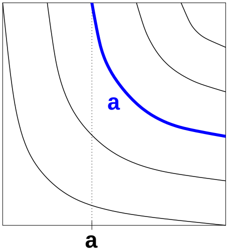
Reconstruction of additive generators

Motivation example

- substitute $y := I_T(x, a)$, $a \in [0, 1]$, in $T'(x, y) = \frac{t'(x)}{t'(T(x,y))}$:
  - then $T(x, y) = a = \text{const.}$

$$T'(x, I_T(x, a)) = \frac{t'(x)}{t'(a)} = \frac{t'(x)}{b}, \quad t'(x) = b \cdot T'(x, I_T(x, a))$$

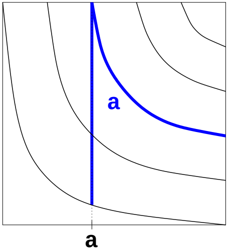
- $b \ldots$ any negative constant, WLOG -1
- applicable if $t'(a) \in]-\infty, 0[$ and $x \in [a, 1]$



Reconstruction of additive generators

- choose $a \in]0, 1]$
- combine the two previous examples

$\Rightarrow$ additive generator t is reconstructed on the whole $[0, 1]$



Reconstruction of additive generators

Lemma

- T … continuous Archimedean t-norm
- $x, y \in [0,1]$, $x \geq y$
- t … additive generator of T
  - with finite derivatives at x, y, and $T(x,y)$
  - $t'(y) \neq 0$
- Then:

$$T'(x, I_T(x,y)) = \frac{t'(x)}{t'(y)}$$

Theorem

- T ... continuous Archimedean t-norm
  - has an additive generator with a finite non-zero derivative in some point $a \in]0,1]$
- Then:

$$t'(x) = \begin{cases} T'(x, I_T(x,a)) & \text{if} \quad x \geq a \\ \frac{1}{T'(a, I_T(a,x))} & \text{if} \quad x < a \end{cases}$$

  - t ... additive generator of T

Corollary

If t is absolutely continous:

$$t(x) = \begin{cases} \int_x^1 T'(z, I_T(z,a))\, \mathrm{d}z & \text{if } x \geq a \\ \int_x^a \frac{1}{T'(a, I_T(a,z))}\, \mathrm{d}z + \int_a^1 T'(z, I_T(z,a))\, \mathrm{d}z & \text{if } x < a \end{cases}$$

Results:

- Reconstruction of multiplicative generators
  - derivatives along “zero line”
- Reconstruction of additive generators
  - derivatives along
    - countour line
    - line parallel with axis
- Alternative to the previous proofs
- Easier to apply
- Shows relation between the shape of a t-norm and the shape of its generators
- Only for given subsets of continuous Archimedean t-norms
  - the subsets are general enough
  - cover all the continuous Archimedean t-norms mentioned in literature

Open problem:

- Formulation of the second method:
- If T is a continuous Archimedean t-norm
  - has an additive generator
    - absolutely continuous
    - with a finite non-zero derivative in some point $a \in]0, 1]$
- Then:
  - an additive generator of T can be reconstructed by the presented method
- The constraints are posed mostly on the generator.
- It would be much more suitable to pose the constraint on the t-norm.