Bell’s inequalities and Pauli matrices

Martin Bohata, Jan Hamhalter

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Outline

*-algebras

- A *-algebra $\mathscr{C}$ is an associative algebra over $\mathbb{C}$ with a map $^* : \mathscr{C} \to \mathscr{C}$ satisfying the following properties:

$$\begin{aligned}(a+b)^* &= a^* + b^*,\\ (\alpha a)^* &= \overline{\alpha} a^*,\\ (ab)^* &= b^* a^*,\\ (a^*)^* &= a,\end{aligned}$$

  for all $a, b \in \mathscr{C}$ and $\alpha \in \mathbb{C}$.
- An element $a \in \mathscr{C}$ is called positive if $a = x^* x$ for some $x \in \mathscr{C}$. An element $a \in \mathscr{C}$ is called self-adjoint if $a = a^*$.
- The partial ordering on $\mathscr{C}$ is given by the positive cone $\mathscr{C}^+ = \{a^* a | a \in \mathscr{C}\}$ (i.e. $a \leq b$ if $b - a \in \mathscr{C}^+$).

*-algebras

- A linear functional φ on a unital *-algebra $\mathscr{C}$ is called state if $\varphi(a^*a) \geq 0$ and $\varphi(\mathbf{1}) = 1$.
- A state φ is called faithful if $\varphi(a^*a) = 0$ implies $a = 0$.
- A state φ is called the trace if $\varphi(ab) = \varphi(ba)$.
- A *-algebra $\mathscr{C}$ is a C*-algebra if $\mathscr{C}$ is a Banach space and $\|a^*a\| = \|a\|^2$.
- A representation of a C*-algebra $\mathscr{C}$ is a *-homomorphism π (i.e homomorphism which preserves *-operation) from $\mathscr{C}$ into the C*-algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators on a Hilbert space $\mathscr{H}$.

GNS construction

- Let $\mathscr{C}$ be a C*-algebra and π be a representation of $\mathscr{C}$. If there is a vector $x \in \mathscr{H}$ for which the linear subspace $\pi(\mathscr{C})x = \{\pi(A)x | A \in \mathscr{C}\}$ is everywhere dense in $\mathscr{H}$, π is described as a cyclic representation and x is termed a cyclic vector for π.

GNS construction

If φ is a state of a C-algebra $\mathscr{C}$, there is a cyclic representation π_φ of $\mathscr{C}$ on a Hilbert space $\mathscr{H}_\varphi$, and a unit cyclic vector x_φ for π_φ such that*

$$\varphi(A) = \langle \pi_\varphi(A)x_\varphi, x_\varphi \rangle$$

for all $A \in \mathscr{C}$. Moreover, the representation is unique up to unitary equivalence.

Bell's inequality (CHSH version)

CHSH form of Bell's inequality

X_1, X_2, Y_1, Y_2 - random variables such that $X_1(x_1), X_2(x_2)$, $Y_1(y_1), Y_2(y_2) \in [-1, 1]$ and $EX_1 = EX_2 = EY_1 = EY_2 = 0$.
Then

$$\frac{1}{2}\left|E(X_1(Y_1 + Y_2)) + E(X_2(Y_1 - Y_2))\right| \leq 1.$$

- Bell's inequality was first studied by John Bell in 1964.
- This inequality has provided a possibility to decide a conflict between classical and quantum theory by an experiment.

Algebraic formulation

CHSH inequality in the algebraic formulation

Suppose that $\mathscr{C}$ is a unital abelian C*-algebra and $\mathscr{A}, \mathscr{B} \subset \mathscr{C}$ are unital C*-subalgebras. Let $a_1, a_2 \in \mathscr{A}$ and $b_1, b_2 \in \mathscr{B}$ be self-adjoint elements such that $-\mathbf{1} \leq a_i \leq \mathbf{1}$ and $-\mathbf{1} \leq b_i \leq \mathbf{1}$, $i = 1, 2$. Then

$$\frac{1}{2} \left| \varphi \left(a_1 \left(b_1 + b_2 \right) + a_2 \left(b_1 - b_2 \right) \right) \right| \leq 1.$$

The case of a C*-algebra

Summers, Werner (1987)

Suppose that $\mathscr{C}$ is a unital C*-algebra and $\mathscr{A}, \mathscr{B} \subset \mathscr{C}$ are unital mutually commuting C*-subalgebras. Let $a_1, a_2 \in \mathscr{A}$ and $b_1, b_2 \in \mathscr{B}$ be self-adjoint elements such that $-\mathbf{1} \leq a_i \leq \mathbf{1}$ and $-\mathbf{1} \leq b_i \leq \mathbf{1}$, $i = 1, 2$. Then

$$\frac{1}{2} \left| \varphi \left(a_1 \left(b_1 + b_2 \right) + a_2 \left(b_1 - b_2 \right) \right) \right| \leq \sqrt{2}.$$

- GNS construction and "Radon-Nikodym theorem" are used in the proof of this theorem.

The case of mutually noncommuting algebras and general correlation dualities?

- We replace φ by a general correlation duality $Q(.,.)$.
- We require that Q is a definite (or indefinite) inner product on a complex linear space X.
- The seminorm $\|.\|_Q$ on a linear space is given by

$$\|x\|_Q = \sqrt{Q(x,x)} \qquad (x \in X).$$

Generalization of Bell's type inequalities

Theorem

If X is a complex linear space, Q is an indefinite inner product on X and $a_1, a_2, b_1, b_2 \in X$ such that $\|a_i\|_Q \leq 1$, $\|b_i\|_Q \leq 1$, $i = 1, 2$, then

$$\frac{1}{2} \left| Q\left(a_1, b_1 + b_2\right) + Q\left(a_2, b_1 - b_2\right) \right| \leq \sqrt{2}.$$

- General linear spaces (not only algebras).
- The proof is based on the Cauchy-Schwarz inequality.

Example: notation

- $\mathscr{C} = \mathscr{B}(\mathscr{H}_2) \otimes \mathscr{B}(\mathscr{H}_2)$, where $\mathscr{H}_2$ is a two-dimensional Hilbert space.
- $\{e_1, e_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is the orthonormal basis of $\mathscr{H}_2$.
- $$\Omega = \frac{e_1 \otimes e_2}{\sqrt{2}} - \frac{e_2 \otimes e_1}{\sqrt{2}}.$$
- Let ω_Ω be the vector state given by the vector Ω (i.e. $\omega_\Omega(a) = \langle a\,\Omega, \Omega\rangle$, $a \in \mathscr{C}$).

Example: properties of ω_Ω

- Since

$$\begin{aligned}\omega_\Omega(a\otimes b) &= \frac{1}{2}\Big[\langle ae_1,e_1\rangle\langle be_2,e_2\rangle-\langle ae_1,e_2\rangle\langle be_2,e_1\rangle\\ &- \langle ae_2,e_1\rangle\langle be_1,e_2\rangle+\langle ae_2,e_2\rangle\langle be_1,e_1\rangle\Big]\end{aligned}$$

then

$$\omega_\Omega(a\otimes\mathbf{1})=\frac{1}{2}\langle ae_1,e_1\rangle+\frac{1}{2}\langle ae_2,e_2\rangle\,,$$

$$\omega_\Omega(\mathbf{1}\otimes b)=\frac{1}{2}\langle be_2,e_2\rangle+\frac{1}{2}\langle be_1,e_1\rangle\,.$$

So the vector state ω_Ω restrict to the trace on the mutually commuting partial subalgebras $\mathscr{A}=\mathscr{B}(\mathscr{H}_2)\otimes\mathbf{1}$, $\mathscr{B}=\mathbf{1}\otimes\mathscr{B}(\mathscr{H}_2)$ of $\mathscr{C}$.

Example: value of ω_Ω on special elements

- Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Assume that $a_1, a_2 \in \mathscr{A}$, $b_1, b_2 \in \mathscr{B}$ are given by

$$\begin{aligned} a_1 &= \sigma_1 \otimes \mathbf{1}, \qquad & b_1 = \mathbf{1} \otimes \frac{\sigma_1 + \sigma_2}{\sqrt{2}}, \\ a_2 &= \sigma_2 \otimes \mathbf{1}, \qquad & b_2 = \mathbf{1} \otimes \frac{\sigma_1 - \sigma_2}{\sqrt{2}}. \end{aligned}$$

- $$\begin{aligned} a_1 (b_1 + b_2) &= \sqrt{2}\, \sigma_1 \otimes \sigma_1, \\ a_2 (b_1 - b_2) &= \sqrt{2}\, \sigma_2 \otimes \sigma_2. \end{aligned}$$

Example: value of ω_Ω on special elements

- Obviously

$$\begin{aligned} \sigma_1 e_1 &= e_2, & \sigma_2 e_1 = i e_2, \\ \sigma_1 e_2 &= e_1, & \sigma_2 e_2 = -i e_1. \end{aligned}$$

- So

$$\begin{aligned} a_1 (b_1 + b_2) \Omega &= \sigma_1 e_1 \otimes \sigma_1 e_2 - \sigma_1 e_2 \otimes \sigma_1 e_1 \\ &= e_2 \otimes e_1 - e_1 \otimes e_2, \\ a_2 (b_1 - b_2) \Omega &= \sigma_2 e_1 \otimes \sigma_2 e_2 - \sigma_2 e_2 \otimes \sigma_2 e_1 \\ &= e_2 \otimes e_1 - e_1 \otimes e_2, \end{aligned}$$

Example: value of ω_Ω on special elements

- Since

$$\begin{aligned}\omega_\Omega\left(a_1\left(b_1+b_2\right)\right) &= \omega_\Omega\left(a_2\left(b_1-b_2\right)\right)\\ &= \frac{1}{\sqrt{2}}\left\langle e_2\otimes e_1-e_1\otimes e_2, e_1\otimes e_2-e_2\otimes e_1\right\rangle\\ &= -\sqrt{2},\end{aligned}$$

we obtain

$$\frac{1}{2}\left|\omega_\Omega\left(a_1\left(b_1+b_2\right)+a_2\left(b_1-b_2\right)\right)\right| = \sqrt{2}.$$

When is the upper bound $\sqrt{2}$ attained?

Theorem

If X is a complex linear space, Q is a definite inner product on X, $a_1, a_2, b_1, b_2 \in X$ such that $\|a_i\|_Q \leq 1$, $\|b_i\|_Q \leq 1$, $i = 1, 2$ and

$$\frac{1}{2}\left|Q(a_1, b_1 + b_2) + Q(a_2, b_1 - b_2)\right| = \sqrt{2},$$

then

$$a_1 = \frac{e^{i\theta}}{\sqrt{2}}(b_1 + b_2), \qquad a_2 = \frac{e^{i\theta}}{\sqrt{2}}(b_1 - b_2),$$

where $\theta \in [0, 2\pi)$.

- Compare this theorem with the above example.

Notation for further results

- Let $\mathscr{C}$ be a unital *-algebra and $\mathscr{A}$, $\mathscr{B}$ be *-subalgebras of $\mathscr{C}$.
- If φ is a state, then φ defines a correlation duality Q given by
$$Q(x,y) = \varphi(y^*x) \qquad x,y \in \mathscr{C}.$$
This case is the most frequently discussed in QFT by Summers, Werner (and others).
- If φ is a faithful state, then $Q(x,y) = \varphi(y^*x)$ defines an inner product.

The main result

Theorem

Suppose that φ *is a faithful state on* $\mathscr{C}$. *Let* $a_1, a_2 \in \mathscr{A}$ *and* $b_1, b_2 \in \mathscr{B}$ *be self-adjoint elements such that* $a_i^2, b_i^2 \leq \mathbf{1}$, $i = 1, 2$. *Assume that*

$$\frac{1}{2}|\varphi(a_1(b_1 + b_2) + a_2(b_1 - b_2))| = \sqrt{2}.$$

Then

$$a_1^2 = a_2^2 = \mathbf{1},$$
$$a_1 a_2 + a_2 a_1 = 0.$$

The same relations hold for b_1, b_2.

Consequences

- a_1, a_2 are Pauli spin matrices.
- $\mathscr{A} \cap \mathscr{B}$ is nontrivial. This intersection excludes independency of algebras in the sense of extended locality (i.e. $\mathscr{A} \cap \mathscr{B} = \mathbb{C}\mathbf{1}$).
- If φ is faithful on $\mathscr{C}$, then $\mathscr{A}$ and $\mathscr{B}$ are not mutually commuting algebras (If $\mathscr{A}$ and $\mathscr{B}$ were mutually commuting algebras then a_1, a_2 would commute and it would be a conflict with relations for a_1, a_2).

Consequences

- The elements a_1, a_2, b_1, b_2 can be chosen such that

$$\frac{1}{2}(a_1(b_1 + b_2) + a_2(b_1 - b_2)) = \sqrt{2}\mathbf{1}.$$

Therefore

$$\psi\left(\frac{1}{2}(a_1(b_1 + b_2) + a_2(b_1 - b_2))\right) = \sqrt{2}$$

for all states ψ. It is known that Bell's inequality holds for product state. Accordingly, ψ cannot be product states.

Is the state φ the trace?

- A state φ is called weakly uncoupled across $\mathscr{A}$ and $\mathscr{B}$ if for all $a_1, a_2 \in \mathscr{A}$ and $b_1, b_2 \in \mathscr{B}$ we have

$$\varphi(a_1 a_2 b_1 b_2) = \varphi(b_1 a_1 a_2 b_2).$$

- If $\mathscr{A}$ and $\mathscr{B}$ are mutually commuting *-subalgebras of a *-algebra $\mathscr{C}$ then any state of $\mathscr{C}$ is weakly uncoupled across $\mathscr{A}$ and $\mathscr{B}$.
- For any weakly uncoupled state we have

$$\varphi(ab) = \varphi(ba),$$

for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$.

Is the state φ the trace?

Theorem

*Let $\mathscr{A}$ and $\mathscr{B}$ be unital *-subalgebras of a unital *-algebra $\mathscr{C}$. Let φ be a weakly uncoupled state on $\mathscr{C}$ across $\mathscr{A}$ and $\mathscr{B}$ which restricts to a faithful state on both $\mathscr{A}$ and $\mathscr{B}$. Let $a_1, a_2 \in \mathscr{A}$ and $b_1, b_2 \in \mathscr{B}$ be self-adjoint elements such that $a_i^2, b_i^2 \leq \mathbf{1}$, $i = 1, 2$. Suppose that*

$$\frac{1}{2}|\varphi(a_1(b_1 + b_2) + a_2(b_1 - b_2))| = \sqrt{2}.$$

Then

$$a_1^2 = a_2^2 = \mathbf{1}, \quad a_1a_2 + a_2a_1 = 0.$$

*The same relations hold for b_1, b_2. Moreover, φ restricts to the trace on the unital *-subalgebras generated by a_1, a_2 and b_1, b_2 respectively.*

Conclusion

- Bell's inequalities were generalized to the complex linear spaces with an indefinite inner product.
- The upper bound $\sqrt{2}$ is attained exactly in the case of Pauli spin matrices.
- The state is the trace on the partial subalgebras in the special case.

References

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