ODE: Practice problems—Analyzing solutions

For the following equations, sketch their vector fields and find stationary solutions (if possible). If an equation is autonomous, determine stability of equilibria.

1. $y' = y^2 - y$;

2. $y' = xy + x$;

3. $y' = \dfrac{(y-1)^2}{e^y - 1}$;

4. $y' = y - \sqrt{x}$;

5. $\dot{x} = x^2 - tx$;

6. $y' = 3y(5 - y)$;

7. $y' = \dfrac{xy - 1}{x}$;

8. $y' = x^2 + y^2 - 1$;

9. $(y^2 - 1)y' = y^2$;

10. $y' = y^2 - x^2$;

11. $y' = \dfrac{y}{1 - y}$;

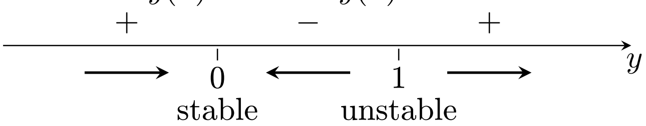
12. $\dot{x} = \dfrac{t + x}{x}$;

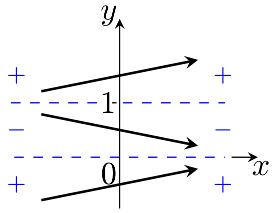
13. $y' = y - y^3$;

14. $y' = x^2 - yx^2$.

Solutions

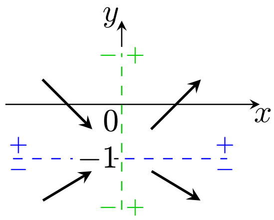
1. $y = y(y-1)$. Stat. sol. $y(x) = 0$ and $y(x) = 1$. It is autonomous.



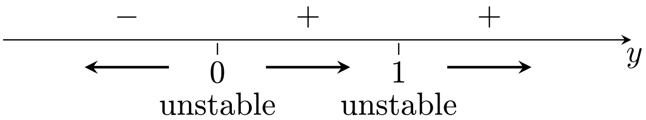


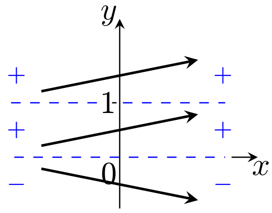
Equilibria: $y_e = 1$ unstable, $y_e = 0$ stable.

2. $y = x(y+1)$. Stat. sol. $y(x) = -1$. It is not autonomous.



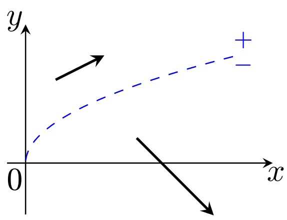
3. $e^y \neq 1 \implies y \neq 0$. Stac. řešení $y(x) = 1$. It is autonomous.



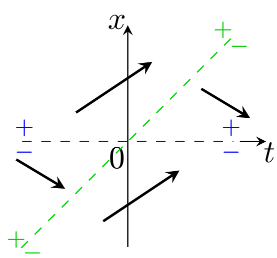


Equilibria: $y_e = 1$ unstable (semistable).
Remark: $y_e = 0$ would be unstable if it were an equilibrium, but it is not (division by zero).

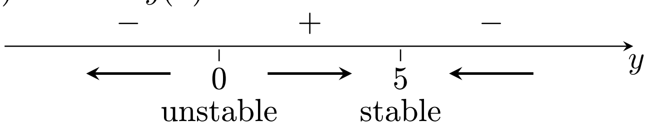
4. $x \geq 0$. Stat. sol. do not exist.
Dividing line for the sign: $y = \sqrt{x}$.
It is not autonomous.

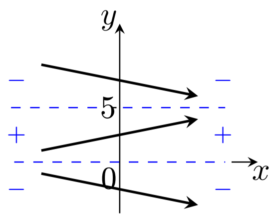


5. $\dot{x} = x(x-t)$. Stat. sol. $x(t) = 0$. It is not autonomous.



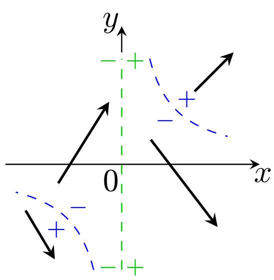
6. Stat. sol. $y(x) = 0$ and $y(x) = 5$. It is autonomous.



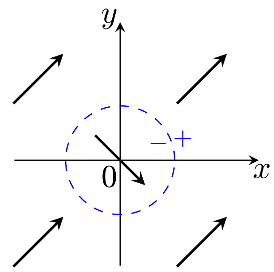


Equilibria: $y_e = 5$ stable, $y_e = 0$ unstable.

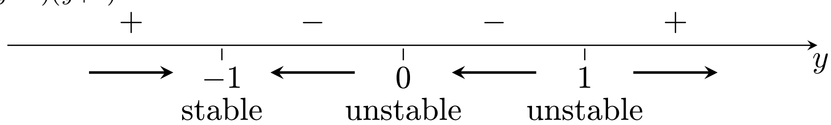
7. $x \neq 0$. Stat. sol. není.
Factors for the sign: x and $xy - 1$. Dividing line: $x = 0$ and $xy = 1$, this is a hyperbola. What are the signs in regions determined by the hyperbola? We substitute $(x, y) = (0, 0)$ and see that the sign is negative "inside", similarly we treat the outside regions.
It is not autonomous.

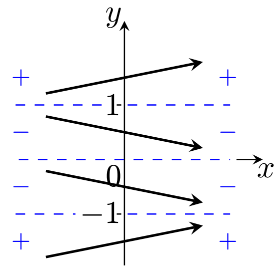


8. Stat. sol. do not exist.
Dividing line for the sign: $x^2 + y^2 = 1$, a circle. The sign inside for instance by substituting $(x, y) = (0, 0)$: $y' < 0$.
It is not autonomous.



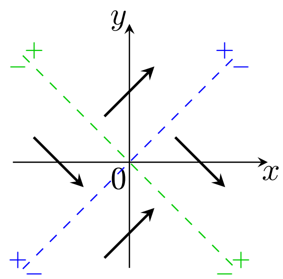
9. $y' = \frac{y^2}{(y-1)(y+1)}$. Stat. sol. $y(x) = 0$. It is autonomous.



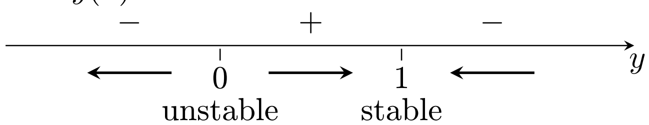


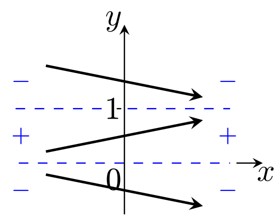
Equilibria: $y_e = 0$ unstable.
Note: $y(x) = 1$ and $y(x) = -1$ are allowed in the equation, but they do not make the derivative zero.

10. $y = (y - x)(y + x)$. Stat. sol. do not exist. It is not autonomous.



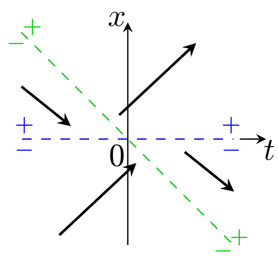
11. $y \neq 1$. Stat. sol. $y(x) = 0$. It is autonomous.



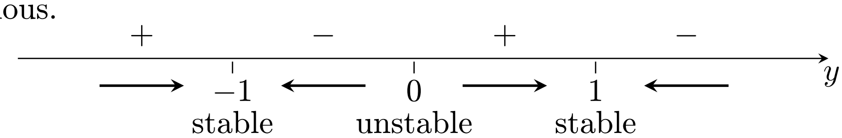


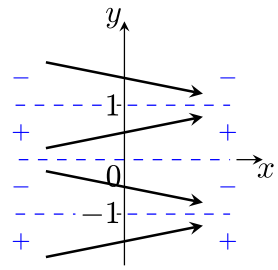
Equilibria: $y_e = 0$ unstable.
Remark: $y_e = 1$ would be stable if it were an equilibrium, but it is not (division by zero). In fact, it was enough to do the part of the picture around $y = 0$.

12. $x \neq 0$. Stat. sol. do not exist. It is not autonomous.



13. $y = (1 + y)y(1 - y)$. Stat. sol. $y(x) = 0$, $y(x) = 1$ and $y(x) = -1$. It is autonomous.





Equilibria: $y_e = 1$ stable, $y_e = 0$ unstable, $y_e = -1$ stable.

14. $y = x^2(1 - y)$, $x^2 \geq 0$ does not influence sign. Stat. sol. $y(x) = 1$. It is not autonomous.