ODE: Practice problems—Method of variation

1. For the equation $y' = \frac{2}{x^3} - \frac{3y}{x}$ solve the following initial value problems:

a) $y(-1) = 3$; b) $y(1) = 1$; c) $y(0) = 3$.

Solve the following initial value problems:

2. $y' = 2y + \frac{1}{\sqrt{x}}e^{2x}$, $y(1) = e^2$;

3. $y' = \frac{2}{x}y + x^2\sin(x)$, $y(\pi) = 2\pi^2$;

4. $y' + y = 13x$, $y(0) = 10$;

5. $x' = \frac{x}{t+1} + 1$, $x(0) = -2$;

6. $y' = 3x^2y - e^{x^3}$, $y(0) = 13$;

7. $y' + \frac{2xy}{x^2-4} = \frac{2x}{(x^2-4)^2}$, $y(1) = -\frac{\ln(3)}{3}$;

8. $x' = 2t^3 - 2tx$, $x(0) = 2$;

9. $y' + \frac{2}{x+1} = \frac{y}{x-1}$, $y(2) = \ln(3)$;

10. $y' + y + x = 0$, $y(0) = 0$;

11. $y' + \frac{y}{x-1} = 6x$, $y(0) = -4$;

12. $xy' + y = \frac{1}{x}$, $y(-1) = 0$;

13. $\dot{x} = x\cot(t) + 2t\sin^2(t)$, $x\left(\frac{7\pi}{2}\right) = 1$.

14. Find a general solution of the equation $y' = \frac{x(y-1)}{x^2+1} + \sqrt{x^2+1}$.

15. Solve the initial value problem $y' = \frac{y(x+1)}{x} - \frac{x+1}{x}$, $y(-1) = 1 - \frac{3}{e}$.

16. Consider the equation $y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 1 + x^2$.
a) Prove that $\{x, x^2 - 1\}$ is its fundamental system.
b) Find a general solution of the associated homogeneous equation.
c) Find a general solution of the given equation.

Solve the following initial value problems:

17. $\ddot{x} + x = \frac{1}{\cos^3(t)}$, $x(\pi) = \dot{x}(\pi) = \frac{1}{2}$;

18. $y'' + 4y' + 4y = \frac{-e^{-2x}}{x^2-1}$, $y(0) = 1$, $y'(0) = -2$.

Find general solutions for the following equations:

19. $x'' + 9x = \frac{9}{\sin(3t)}$;

20. $y'' + 2y' + y = 15e^{-x}\sqrt{x}$;

21. $\ddot{x} + 4x = 8\sin^2(2t)$;

22. $\ddot{x} + 4x = -8\cot(2t)$.

Solutions

1. Conditions: $x \neq 0$. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $y' = -\frac{3y}{x}$, so $\int \frac{dy}{y} = -3\int \frac{dx}{x}$, $\ln|y| = -3\ln|x| + c = \ln\big(\frac{1}{|x^3|}\big) + c$, $y = \pm e^c \frac{1}{x^3}$, general homogeneous solution is $y_h(x) = \frac{C}{x^3}$, $x \neq 0$.
Variation: $y(x) = \frac{C(x)}{x^3}$, then $\frac{C'(x)}{x^3} = \frac{2}{x^3}$, $C'(x) = 2$. This yields $C(x) = 2x$ or $C(x) = 2x + C$, general solution is $y(x) = \frac{2x+C}{x^3}$, $x \in (-\infty, 0)$ or $x \in (0, \infty)$.
Remark: The guessing method does not help, the equation does not have constant coefficients and besides, we cannot guess $\frac{2}{x^3}$.
Initial values: a) $C = -1$, so $y_a(x) = \frac{2x-1}{x^3}$, $x \in (-\infty, 0)$;
b) $C = -1$, so $y_b(x) = \frac{2x-1}{x^3}$, $x \in (0, \infty)$; c) $y_c(x)$ neex.

2. Conditions: $x > 0$. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $y' = 2y$, rewrite as $y' - 2y = 0$, constant coefficients, hence we can use $\lambda - 2 = 0$, then $\lambda = 2$, general homogeneous solution is $y_h(x) = Ce^{2x}$, $x \in I\!R$.
Separation is also possible: $\int \frac{dy}{y} = \int 2dx$, $\ln|y| = 2x + c$, $y = \pm e^c e^{2x}$, $y_h(x) = Ce^{2x}$.
Variation: $y(x) = C(x)e^{2x}$, then $C'(x)e^{2x} = \frac{1}{\sqrt{x}}e^{2x}$, $C'(x) = \frac{1}{\sqrt{x}}$. This yields $C(x) = 2\sqrt{x}$ or $C(x) = 2\sqrt{x} + C$, general solution is $y(x) = 2\sqrt{x}e^{2x} + Ce^{2x}$, $x \in (0, \infty, 0)$.
Remark: The guessing method does not help, the equation does have constant coefficients, but $\frac{1}{\sqrt{x}}e^{2x}$.
Initial value: $C = -1$, so $y(x) = 2\sqrt{x}e^{2x} - e^{2x}$, $x \in (0, \infty, 0)$.

3. Conditions: $x \neq 0$. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $y' = \frac{2}{x}y$, so $\int \frac{dy}{y} = 2\int \frac{dx}{x}$, $\ln|y| = 2\ln|x| + c = \ln(|x^2|) + c$, $y = \pm e^c x^2$, general homogeneous solution is $y_h(x) = Cx^2$, $x \neq 0$.
Variation: $y(x) = C(x)x^2$, then $C'(x)x^2 = x^2\sin(x)$, $C'(x) = \sin(x)$. This yields $C(x) = -\cos(x)$ or $C(x) = C - \cos(x)$, general solution is $y(x) = Cx^2 - x^2\cos(x)$, $x \in (-\infty, 0)$ or $x \in (0, \infty)$.
Remark: The guessing method does not help, the equation does not have constant coefficients.
Initial value: $C = 1$, so $y(x) = x^2(1 - \cos(x))$, $x \in (0, \infty)$.

4. Conditions: none. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $y' + y = 0$, constant coefficients, hence we can use $\lambda + 1 = 0$, then $\lambda = -1$, general homogeneous solution is $y_h(x) = Ce^{-x}$, $x \in I\!R$.
Separation is also possible: $\int \frac{dy}{y} = -\int 1dx$, $\ln|y| = -x + c$, $y = \pm e^c e^{-x}$, $y_h(x) = Ce^{-x}$.
Variation: $y(x) = C(x)e^{-x}$, then $C'(x)e^{-x} = 13x$, $C'(x) = 13x\,e^x$. This yields $C(x) = 13x\,e^x - 13e^x$ or $C(x) = 13x\,e^x - 13e^x + C$, general solution is $y(x) = 13x - 13 + Ce^{-x}$, $x \in I\!R$.
Remark: This equation can be also solved by the guessing method. Guess $y_p = Ax + B$ leads to $A = 13$, $B = -13$.
Initial value: $C = 23$, so $y(x) = 13x - 13 + 23e^{-x}$, $x \in I\!R$.

5. Conditions: $t \neq -1$. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $x' = \frac{x}{t+1}$, so $\int \frac{dx}{x} = \int \frac{dt}{t+1}$, $\ln|x| = \ln|t+1| + c$, $x = \pm e^c(t+1)$, general homogeneous solution is $x_h(t) = C(t+1)$, $t \neq -1$.
Variation: $x(t) = C(t)(t+1)$, then $C'(t)(t+1) = 1$, $C'(t) = \frac{1}{t+1}$. This yields $C(t) = \ln|t+1|$ or $C(t) = \ln|t+1| + C$, general solution is $x(t) = C(t+1) + \ln|t+1|(t+1)$, $t \in (-\infty, -1)$ or $t \in (-1, \infty)$.

Remark: The guessing method does not help, the equation does not have constant coefficients.
Initial value: $C=-2$, so $x(t)=\ln|t+1|(t+1)-2(t+1)$, $t\in(-1,\infty)$.

6. Conditions: none. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $y'=3x^2y$, so $\int\frac{dy}{y}=3\int x^2\,dx$, $\ln|y|=x^3+c$, $y=\pm e^ce^{x^3}$, general homogeneous solution is $y_h(x)=Ce^{x^3}$, $x\in I\!R$.
Variation: $y(x)=C(x)e^{x^3}$, then $C'(x)e^{x^3}=-e^{x^3}$, $C'(x)=-1$. This yields $C(x)=-x$ or $C(x)=C-x$, general solution is $y(x)=Ce^{x^3}-x\,e^{x^3}$, $x\in I\!R$.
Remark: The guessing method does not help, the equation does not have constant coefficients and besides, we cannot guess e^{x^3}.
Initial value: $C=13$, so $y(x)=(13-x)e^{x^3}$, $x\in I\!R$.

7. Conditions: $x\neq\pm2$. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $y'+\frac{2xy}{x^2-4}=0$, so $\int\frac{dy}{y}=-\int\frac{2x}{x^2-4}\,dx$, $\ln|y|=-\ln|x^2-4|+c=\ln|(x^2-4)^{-1}|+c$, $y=\pm e^c\frac{1}{x^2-4}$, general homogeneous solution is $y_h(x)=C\frac{1}{x^2-4}$, $x\neq\pm2$.
Variation: $y(x)=C(x)\frac{1}{x^2-4}$, then $C'(x)\frac{1}{x^2-4}=\frac{2x}{(x^2-4)^2}$, $C'(x)=\frac{2x}{x^2-4}$. This yields $C(x)=\ln|x^2-4|$ or $C(x)=\ln|x^2-4|+C$, general solution is $y(x)=\frac{C}{x^2-4}+\frac{\ln|x^2-4|}{x^2-4}$, $x\neq\pm2$.
Remark: The guessing method does not help, the equation does not have constant coefficients and besides, we cannot guess $\frac{2x}{(x^2-4)^2}$.
Initial value: $C=0$, so $y(x)=\frac{\ln|x^2-4|}{x^2-4}=\frac{\ln(4-x^2)}{x^2-4}$, $x\in(-2,2)$.

8. Conditions: none. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $x'=-2tx$, so $\int\frac{dx}{x}=-\int 2t\,dt$, $\ln|x|=-t^2+c$, $x=\pm e^ce^{-t^2}$, general homogeneous solution is $x_h(t)=Ce^{-t^2}$, $t\in I\!R$.
Variation: $y(x)=C(t)e^{-t^2}$, then $C'(t)e^{-t^2}=2t^3$, $C'(t)=2t^3e^{t^2}$. This yields $C(x)=t^2e^{t^2}-e^{t^2}$ or $C(x)=t^2e^{t^2}-e^{t^2}+C$, general solution is $y(x)=t^2-1+Ce^{-t^2}$, $t\in I\!R$.
Remark: The guessing method does not help, the equation does not have constant coefficients.
Initial value: $C=3$, so $x(t)=t^2-1+3e^{-t^2}$, $t\in I\!R$.

9. Conditions: $x\neq\pm1$. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $y'=\frac{y}{x-1}$, so $\int\frac{dy}{y}=\int\frac{dx}{x-1}$, $\ln|y|=\ln|x-1|+c$, $y=\pm e^c(x-1)$, general homogeneous solution is $y_h(x)=C(x-1)$, $x\neq1$.
Variation: $y(x)=C(x)(x-1)$, then $C'(x)(x-1)=-\frac{2}{x+1}$, $C'(x)=-\frac{2}{(x-1)(x+1)}=\frac{1}{x+1}-\frac{1}{x-1}$. This yields $C(x)=\ln|x+1|-\ln|x-1|=\ln\left|\frac{x+1}{x-1}\right|$ or $C(x)=\ln\left|\frac{x+1}{x-1}\right|+C$, general solution is $y(x)=\ln\left|\frac{x+1}{x-1}\right|(x-1)+C(x-1)$, $x\neq\pm1$.
Remark: The guessing method does not help, the equation does not have constant coefficients and besides, we cannot guess $\frac{2}{x+1}$.
Initial value: $C=0$, so $y(x)=\ln\left|\frac{x+1}{x-1}\right|(x-1)$, $x\in(1,\infty)$.

10. Conditions: none. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible. Careful, the proper form is $y'+y=-x$, so $b(x)=-x$.
Homogeneous case first, $y'=-y$, so $\int\frac{dy}{y}=-\int dx$, $\ln|y|=-x+c$, $y=\pm e^ce^{-x}$, general homogeneous solution is $y_h(x)=Ce^{-x}$, $x\in I\!R$.
Variation: $y(x)=C(x)e^{-x}$, then $C'(x)e^{-x}=b(x)=-x$, $C'(x)=-x\,e^x$. Integration bz parts yields $C(x)=-x\,e^x+e^x$ or $C(x)=-x\,e^x+e^x+C$, general solution is
$y(x)=(e^x-x\,e^x+C)e^{-x}=1-x+Ce^{-x}$, $x\in I\!R$.
Remark: This equation can be also solved by the guessing method. Guess $y_p=Ax+B$ leads to $A=-1$, $B=1$.

Initial value: $C=-1$, so $y(x)=1-x-e^{-x}$, $x\in I\!R$.

11. Conditions: $x\neq 1$. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $y'=-\frac{y}{x-1}$, so $\int\frac{dy}{y}=-\int\frac{dx}{x-1}$, $\ln|y|=-\ln|x-1|+c$, $y=\pm e^c\frac{1}{x-1}$, general homogeneous solution is $y_h(x)=\frac{C}{x-1}$, $x\neq 1$.
Variation: $y(x)=\frac{C(x)}{x-1}$, then $\frac{C'(x)}{x-1}=6x$, $C'(x)=6x^2-6x$. This yields $C(x)=2x^3-3x^2$ or $C(x)=2x^3-3x^2+C$, general solution is $y(x)=\frac{2x^3-3x^2+C}{x-1}$, $x\neq 1$.
Remark: The guessing method does not help, the equation does not have constant coefficients.
Initial value: $C=4$, so $y(x)=\frac{2x^3-3x^2+4}{x-1}$, $x\in(\infty,1)$.

12. Conditions: $x\neq 0$. Method: separation not possible, but it can be written as non-homogeneous linear equation $y'+\frac{1}{x}y=\frac{1}{x^2}$, so variation is possible.
Homogeneous case first, $y'=-\frac{y}{x}$, so $\int\frac{dy}{y}=-\int\frac{dx}{x}$, $\ln|y|=-\ln|x|+c=\ln\bigl(\frac{1}{|x|}\bigr)+c$, $y=\pm e^c\frac{1}{x}$, general homogeneous solution is $y_h(x)=\frac{C}{x}$, $x\neq 0$.
Variation: $y(x)=\frac{C(x)}{x}$, then $\frac{C'(x)}{x}=\frac{1}{x^2}$, $C'(x)=\frac{1}{x}$. This yields $C(x)=\ln|x|$ or $C(x)=\ln|x|+C$, general solution is $y(x)=\frac{C}{x}+\frac{\ln|x|}{x}$, $x\in(-\infty,0)$ or $x\in(0,\infty)$.
Remark: The guessing method does not help, the equation does not have constant coefficients and besides, we cannot guess $\frac{1}{x^2}$.
Initial value: $C=0$, so $y(x)=\frac{\ln|x|}{x}=\frac{\ln(-x)}{x}$, $x\in(-\infty,0)$.

13. Conditions: $t\neq k\pi$. Method: separation not possible, but it is a non-homogeneous linear equation, so variation possible.
Homogeneous case first, $\dot{x}=x\cot(t)$, so $\int\frac{dx}{x}=\int\frac{\cos(t)dt}{\sin(t)}$, substitution $z=\sin(t)$, $\ln|x|=\ln|\sin(t)|+c$, $y=\pm e^c\sin(t)$, general homogeneous solution is $x_h(t)=C\sin(t)$, $t\neq k\pi$.
Variation: $x(t)=C(t)\sin(t)$, then $C'(t)\sin(t)=2t\sin^2(t)$, $C'(t)=2t\sin(t)$. This yields pomoc per-partes $C(x)=-2t\cos(t)+2\sin(t)$ or $C(x)=-2t\cos(t)+2\sin(t)+C$, general solution is $x(t)=2\sin^2(t)-2t\sin(t)\cos(t)+C\sin(t)=2\sin^2(t)-t\sin(2t)+C\sin(t)$, $t\in(k\pi,(k+1)\pi)$ pro $k\in Z\!\!Z$.
Remark: The guessing method does not help, the equation does not have constant coefficients and besides, we cannot guess $2t\sin^2(t)$.
Initial value: $C=1$, so $y(x)=2\sin^2(t)-t\sin(2t)+\sin(t)$, $t\in(3\pi,4\pi)$.

14. Conditions: none. Method: separation not possible, the equation is not formally linear, but it can be written as one:

$$y'=\frac{x}{x^2+1}y-\frac{x}{x^2+1}+\sqrt{x^2+1}.$$

Homogeneous case $y'=\frac{x}{x^2+1}y$: $\int\frac{dy}{y}=\int\frac{x}{x^2+1}$, substitution $z=x^2+1$, $\ln|y|=\frac{1}{2}\ln|x^2+1|+c=\ln\sqrt{x^2+1}+c$, $y=\pm e^c\sqrt{x^2+1}$, general homogeneous solution is $y_h(x)=C\sqrt{x^2+1}$, $x\in I\!R$.
Variation: $y(x)=C(x)\sqrt{x^2+1}$, then $C(x)'\sqrt{x^2+1}=-\frac{x}{x^2+1}+\sqrt{x^2+1}$, hence

$$C(x)=\int-\frac{x}{(x^2+1)\sqrt{x^2+1}}+1\,dx=-\int\frac{x}{(x^2+1)\sqrt{x^2+1}}\,dx+x=\left|\begin{array}{l} z=\sqrt{x^2+1}\\ dz=\frac{x}{\sqrt{x^2+1}}dx\end{array}\right|$$
$$=x-\int\frac{dz}{z^2}=x+\frac{1}{z}+C=x+\frac{1}{\sqrt{x^2+1}}+C,$$

so general solution is $y(x)=x\sqrt{x^2+1}+1+C\sqrt{x^2+1}$, $x\in I\!R$.
Remark: The guessing method does not help, the equation does not have constant coefficients and besides, we cannot guess for the RHS.

Alternative: The variation method also works for the type $y'+a(x)(Ay+B)=b(x)$, so we can solve directly: Continuity everywhere, so there is a solution on $I\!R$. Homogeneous equation $y'=\frac{x}{x^2+1}(y-1)$ yields (by separation) $y_h(x)=C\sqrt{x^2+1}+1$, $x\in I\!R$ (this also includes the stationary solution $y(x)=1$). Variation: $y(x)=C(x)\sqrt{x^2+1}+1$, then $y'(x)=$

$C'(x)\sqrt{x^2+1}+C(x)\frac{x}{\sqrt{x^2+1}}$, substituting into the equation and cancelling we get $C'(x)=1$, so $C(x)=x+C$. General solution $y(x)=(x+C)\sqrt{x^2+1}+1$, $x\in I\!R$.
Warning! Due to that $Ay+B$ we really do have to substitute $C(x)\sqrt{x^2+1}+1$ into the equaqtion this time, the trick $C'(x)\sqrt{x^2+1}+1=\sqrt{x^2+1}$ does not work here.

15. Conditions: $x\neq 0$. Method: it is a non-homogeneous linear equation, so variation is possible.
Homogeneous case $y'=\frac{y(x+1)}{x}$: $\int\frac{dy}{y}=\int\frac{x+1}{x}\,dx=\int 1+\frac{1}{x}\,dx$, $\ln|y|=x+\ln|x|+c$, $y=\pm e^c x\,e^x$, general homogeneous solution is $y_h(x)=Cx\,e^x$, $x\neq 0$.
Variation: $y(x)=C(x)x\,e^x$, then $C'(x)x\,e^x=\frac{x+1}{x}$, $C(x)=\int\frac{x+1}{x^2}e^{-x}dx$. We do not know how to integrate this.
Another option? Put the original terms together, use separation: $y'=(y-1)\frac{(x+1)}{x}$, condition $x\neq 0$. We separate and integrate:

$$\int\frac{dy}{y-1}=\int\frac{x+1}{x}\,dx=\int 1+\frac{1}{x}\,dx,$$

so $\ln|y-1|=x+\ln|x|+c$, $y-1=\pm e^c x\,e^x$, hence general solution $y(x)=Cx\,e^x+1$, $x\neq 0$; the choice $C=0$ includes the stationary solution.
Initial value: $C=3$, so $y(x)=3x\,e^x+1$, $x\in(-\infty,0)$.

16. a) Substituting into the associated homogeneous equation we confirm that both functions solve it on $I\!R$. Indeed, for any $x\in I\!R$ we have
$y''-\frac{2x}{1+x^2}y'+\frac{2}{1+x^2}y=[x]''-\frac{2x}{1+x^2}[x]'+\frac{2}{1+x^2}x=0-\frac{2x}{1+x^2}+\frac{2x}{1+x^2}=0.$
$y''-\frac{2x}{1+x^2}y'+\frac{2}{1+x^2}y=[x^2-1]''-\frac{2x}{1+x^2}[x^2-1]'+\frac{2}{1+x^2}[x^2-1]=2-\frac{4x^2}{1+x^2}+\frac{2x^2-2}{1+x^2}=0.$
So we have two solutions and the space of solutions of a second order homogeneous ODE has dimension 2. Thus to confirm a basis it is enough to show that the two functions are linearly independent. We use the Wronskian for that.

$$W(x)=\begin{vmatrix} x & x^2-1\\ [x]' & [x^2-1]'\end{vmatrix}=\begin{vmatrix} x & x^2-1\\ 1 & 2x\end{vmatrix}=2x^2-(x^2-1)=x^2+1\neq 0.$$

Since $\{x,x^2-1\}$ is a basis of the space of all solutions of the associated homogeneous equation, it is also a fundamental system of the given equation.
b) $y_h(x)=ax+b(x^2-1)$, $x\in I\!R$.
c) The equation is linear, the right-hand side is special, but the equation does not have constant coefficients and this rules out the guessing method (undetermined oefficients). We are left with the variation method, so we consider $y(x)=a(x)x+b(x)(x^2-1)$ and set up the equations:

$$\begin{aligned} a'(x)x+b'(x)(x^2-1)&=0\\ a'(x)[x]'+b'(x)[x^2-1]'&=1+x^2\end{aligned}\quad\Longrightarrow\quad\begin{aligned} a'(x)x+b'(x)(x^2-1)&=0\\ a'(x)+b'(x)2x&=1+x^2\end{aligned}$$

From this (for instance by the Cramer rule) $D=x^2+1$, $D_{a'}=-(x^2-1)(1+x^2)$, $D_{b'}=x(1+x^2)$, hence

$$\begin{aligned} a'(x)&=1-x^2\\ b'(x)&=x\end{aligned}\quad\Longrightarrow\quad\begin{aligned} a(x)&=x-\tfrac{1}{3}x^3\\ b(x)&=\tfrac{1}{2}x^2\end{aligned}$$

We obtained a particular solution $y_p(x)=\big(x-\frac{1}{3}x^3\big)x+\frac{1}{2}x^2(x^2-1)=\frac{1}{2}x^2+\frac{1}{6}x^4$ of the given equation, a general solution is given by the formula y_p+y_h, hence we get
$y(x)=\frac{1}{2}x^2+\frac{1}{6}x^4+ax+b(x^2-1)$, $x\in I\!R$.
Remark: We can also find a general solution directly using $a(x)=x-\frac{1}{3}x^3+a$, $b(x)=\frac{1}{2}x^2+b$.

17. The left-hand side is linear with constant coefficients. Char. pol. $p(\lambda)=\lambda^2+1$, char. numbers $\lambda=\pm j$; fund. syst. $\{\sin(t),\cos(t)\}$; general homogeneous solution is $x_h(t)=a\sin(t)+b\cos(t)$, $t\in I\!R$.
The right-hand side is not special, hence variation method: $x(t)=a(t)\sin(t)+b(t)\cos(t)$, equations

$$\begin{matrix} a'(t)\sin(t)+b'(t)\cos(t)=0 \\ a'(t)\cos(t)-b'(t)\sin(t)=\frac{1}{\cos^3(t)} \end{matrix} \implies \begin{matrix} a'(t)=\frac{1}{\cos^2(t)} \\ b'(t)=\frac{-\sin(t)}{\cos^3(t)} \end{matrix} \implies \begin{matrix} a(t)=\tan(t) \\ b(t)=-\frac{1}{2}\frac{1}{\cos^2(t)} \end{matrix}$$

($b(t)$ is done using substitution $z=\cos(t)$).
Thus $x_p(t)=\tan(t)\sin(t)-\frac{1}{2}\frac{1}{\cos^2(t)}\cos(t)$ and $x=x_p+x_h$. A general solution is
$x(t)=\frac{\sin(t)}{\cos^2(t)}-\frac{1}{2}\frac{1}{\cos(t)}+a\sin(t)+b\cos(t)$, $t\neq\frac{\pi}{2}+k\pi$.
Init. conditions: $x(t)=\frac{\sin(t)}{\cos^2(t)}-\frac{1}{2}\frac{1}{\cos(t)}-\sin(t)$, $t\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$.

18. The left-hand side is linear with constant coefficients. Char. pol. $p(\lambda)=\lambda^2+4\lambda+4$, char. number $\lambda=-2$ ($2\times$); fund. syst. $\{e^{-2x},x\,e^{-2x}\}$; general homogeneous solution is
$y_h(x)=ae^{-2x}+bx\,e^{-2x}$, $x\in I\!R$.
The right-hand side is not special, hence variation method: $y(x)=a(x)e^{-2x}+b(x)x\,e^{-2x}$, equations

$$\begin{matrix} a'(x)e^{-2x}+b'(x)x\,e^{-2x}=0 \\ -2a'(x)e^{-2x}+b'(x)(1-2x)e^{-2x}=\frac{e^{-2x}}{x^2-1} \end{matrix} \implies \begin{matrix} a'(x)=\frac{x}{x^2-1} \\ b'(x)=\frac{-1}{x^2-1} \end{matrix} \implies \begin{matrix} a(x)=\frac{1}{2}\ln|x^2-1| \\ b(x)=-\frac{1}{2}\ln|\frac{x-1}{x+1}| \end{matrix}$$

($a(x)$ is done using substitution $z=x^2-1$, $b(x)$ using partial fractions).
Thus $y_p(x)=\frac{1}{2}\ln|x^2-1|\,e^{-2x}-\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|x\,e^{-2x}$ and $y=y_p+y_h$. A general solution is
$y(x)=\frac{1}{2}\ln|x^2-1|e^{-2x}-\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|x\,e^{-2x}+ae^{-2x}+bx\,e^{-2x}$, $x\neq\pm1$.
Init. conditions: $y(x)=\frac{1}{2}\ln|x^2-1|e^{-2x}-\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|x\,e^{-2x}+e^{-2x}$, $x\in(-1,1)$.

19. The left-hand side is linear with constant coefficients. Char. pol. $p(\lambda)=\lambda^2+9$, char. numbers $\lambda=\pm3j$; fund. syst. $\{\sin(3t),\cos(3t)\}$; general homogeneous solution is
$x_h(t)=a\sin(3t)+b\cos(3t)$, $t\in I\!R$.
The right-hand side is not special, hence variation method: $x(t)=a(t)\sin(3t)+b(t)\cos(3t)$, equations

$$\begin{matrix} a'(t)\sin(3t)+b'(t)\cos(3t)=0 \\ 3a'(t)\cos(3t)-3b'(t)\sin(3t)=\frac{9}{\sin(3t)} \end{matrix} \implies \begin{matrix} a'(t)=\frac{3\cos(3t)}{\sin(3t)} \\ b'(t)=-3 \end{matrix} \implies \begin{matrix} a(t)=\ln|\sin(3t)| \\ b(t)=-3t \end{matrix}$$

($a(t)$ is done using substitution $z=\sin(3t)$).
Thus $x_p(t)=\ln|\sin(3t)|\sin(3t)-3t\cos(3t)$ and $x=x_p+x_h$. A general solution is
$x(t)=\ln|\sin(3t)|\sin(3t)-3t\cos(3t)+a\sin(3t)+b\cos(3t)$, $t\neq k\pi$.

20. The left-hand side is linear with constant coefficients. Char. pol. $p(\lambda)=\lambda^2+2\lambda+1$, char. number $\lambda=-1$ ($2\times$); fund. syst. $\{e^{-x},x\,e^{-x}\}$; general homogeneous solution is
$y_h(x)=ae^{-x}+bx\,e^{-x}$, $x\in I\!R$.
The right-hand side is not special, hence variation method: $y(x)=a(x)e^{-x}+b(x)x\,e^{-x}$, equations

$$\begin{matrix} a'(x)e^{-x}+b'(x)x\,e^{-x}=0 \\ -a'(x)e^{-x}+b'(x)(1-x)e^{-x}=15e^{-x}\sqrt{x} \end{matrix} \implies \begin{matrix} a'(x)=-x\sqrt{x}=-15x^{3/2} \\ b'(x)=15\sqrt{x} \end{matrix} \implies \begin{matrix} a(x)=-6x^{5/2} \\ b(x)=10x^{3/2} \end{matrix}$$

Thus $y_p(x)=-6\sqrt{x^5}\,e^{-x}+10\sqrt{x^3}x\,e^{-x}$ and $y=y_p+y_h$. A general solution is
$y(x)=4\sqrt{x^5}e^{-x}+ae^{-x}+bx\,e^{-x}$, $x\in(0,\infty)$.

21. The left-hand side is linear with constant coefficients. Char. pol. $p(\lambda)=\lambda^2+4$, char. numbers $\lambda=\pm2j$; fund. syst. $\{\sin(2t),\cos(2t)\}$; general homogeneous solution is
$x_h(t)=a\sin(2t)+b\cos(2t)$, $t\in I\!R$.
The right-hand side is not special, hence variation method: $x(t)=a(t)\sin(2t)+b(t)\cos(2t)$, equations

$$\begin{matrix} a'(t)\sin(2t)+b'(t)\cos(2t)=0 \\ 2a'(t)\cos(2t)-2b'(t)\sin(2t)=8\sin^2(2t) \end{matrix} \implies \begin{matrix} a'(t)=2\sin^2(2t)\cos(2t) \\ b'(t)=-2\sin^3(2t) \end{matrix}$$

$$\implies \begin{matrix} a(t)=\frac{1}{3}\sin^3(2t) \\ b(t)=\cos(2t)-\frac{1}{3}\cos^3(2t) \end{matrix}$$

($b(t)$ is done using $\int(\cos^2(2t)-1)2\sin(2t)\,dt$ with substitution $z=\cos(2t)$).

Thus $x_p(t) = \frac{1}{3}\sin^3(2t)\sin(2t) + \big(\cos(2t) - \frac{1}{3}\cos^3(2t)\big)\cos(2t)$ and $x = x_p + x_h$. A general solution is

$$x(t) = \tfrac{1}{3}\big(\sin^4(2t) - \cos^4(2t)\big) + \cos^2(2t) + a\sin(3t) + b\cos(3t)$$
$$= \tfrac{1}{3}\big(\sin^2(2t) + \cos^2(2t)\big)\big(\sin^2(2t) - \cos^2(2t)\big) + \cos^2(2t) + a\sin(3t) + b\cos(3t)$$
$$= \tfrac{1}{3}\big(1 - 2\cos^2(2t)\big) + \cos^2(2t) + a\sin(3t) + b\cos(3t) = \tfrac{1}{3} + \tfrac{1}{3}\cos^2(2t) + a\sin(3t) + b\cos(3t),$$

$t \in I\!R$.

Alternative: Rewrite: $\ddot{x} + 4x = 4(1 - \cos(4t))$, the right-hand side is special, namely a combination of two special sides.

1) 4: $d = 0$; $\alpha = 0$ and $\beta = 0$, multiplicity of $\alpha + \beta j = 0$ as a char. number is $m = 0$; hence $x_1(t) = A$.

2) $-4\cos(4t) = e^{0\cdot t}[0\sin(4\cdot t) + (-4)\cos(4\cdot t)]$: $d = 0$; $\alpha = 0$ and $\beta = 4$, multiplicity of $\alpha + \beta j = 4j$ as a char. number is $m = 0$; hence $x_2(t) = t^0 e^0[B\sin(4t) + C\cos(4t)]$.

We guess the form of particular solution $x_p(t) = x_1(t) + x_2(t) = A + B\sin(4t) + C\cos(4t)$, substituting into the given equation we get

$$4A - 12B\sin(4t) - 13C\cos(4t) = 4 - 4\cos(4t), \text{ hence } A = 1,\ B = 0,\ C = \tfrac{1}{3},$$

general solution is $x(t) = x_p(t) + x_h(t) = \frac{1}{3} + \frac{1}{3}\cos^2(2t) + a\sin(3t) + b\cos(3t)$, $t \in I\!R$.

22. The left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 + 4$, char. numbers $\lambda = \pm 2j$; fund. syst. $\{\sin(2t), \cos(2t)\}$; general homogeneous solution is $x_h(t) = a\sin(2t) + b\cos(2t)$, $t \in I\!R$.

The right-hand side is not special, hence variation method: $x(t) = a(t)\sin(2t) + b(t)\cos(2t)$, equations

$$\begin{aligned} a'(t)\sin(2t) + b'(t)\cos(2t) &= 0 \\ 2a'(t)\cos(2t) - 2b'(t)\sin(2t) &= -8\cot(2t) \end{aligned} \implies \begin{aligned} a'(t) &= -2\cot(2t)\cos(2t) \\ b'(t) &= 2\cos(2t) \end{aligned}$$

$$\implies \begin{aligned} a(t) &= \tfrac{1}{2}\ln\Big|\tfrac{\cos(2t)-1}{\cos(2t)+1}\Big| \\ b(t) &= \sin(2t) \end{aligned}$$

$a(t)$ is done using $\int -2\frac{\cos^2(2t)}{\sin(2t)}\,dt = \int -\frac{\cos^2(2t)}{\sin^2(2t)}2\sin(2t)\,dt = \int \frac{\cos^2(2t)}{\cos^2(2t)-1}2\sin(2t)\,dt$, subst. $z = \cos(2t)$, $\int \frac{dz}{z^2-1} = \frac{1}{2}\int \frac{1}{z-1} - \frac{1}{z+1}\,dz = \frac{1}{2}\ln\big|\frac{z-1}{z+1}\big|$.

Thus $x_p(t) = \frac{1}{2}\ln\big|\frac{\cos(2t)-1}{\cos(2t)+1}\big|\sin(2t) + \sin(2t)\cos(2t)$ and $x = x_p + x_h$. A general solution is $x(t) = \frac{1}{2}\ln\big|\frac{\cos(2t)-1}{\cos(2t)+1}\big|\sin(2t) + \sin(2t)\cos(2t) + a\sin(3t) + b\cos(3t)$, $t \neq \frac{\pi}{2}k$.