ODE: Practice problems—Homogeneous systems of equations

For the following systems of equations find their general solutions. Use the matrix approach, but if you want, you can also try elimination for fun.
For all systems also determine stability of the trivial stationary solution $y_1(x) = y_2(x) = 0$.
For 2×2 systems then discuss typical asymptotic behaviour of the general solution at infinity and find the particular solution determined by the given initial conditions.

1. $y_1' = -2y_1 + 4y_2$, $y_2' = y_1 + y_2$ $\qquad y_1(0) = 4,\ y_2(0) = -1$;

2. $y_1' = 2y_1 + y_2$, $y_2' = y_1 + 2y_2$ $\qquad y_1(0) = 3,\ y_2(0) = 1$;

3. $y_1' = y_1 - 3y_2$, $y_2' = 3y_1 + y_2$ $\qquad y_1(0) = 1,\ y_2(0) = 1$;

4. $y_1' = 2y_1 - 3y_2$, $y_2' = 3y_1 - 4y_2$ $\qquad y_1(0) = 2,\ y_2(0) = 1$;

5. $y_1' = y_1 + 4y_2$, $y_2' = 3y_1 + 2y_2$ $\qquad y_1(0) = 3,\ y_2(0) = -4$;

6. $\dot{x}_1 = x_1 - x_2$, $\dot{x}_2 = 2x_1 - x_2$ $\qquad x_1(\pi) = -1,\ x_2(\pi) = 0$;

7. $\dot{x}_1 = 2x_1 + x_2$, $\dot{x}_2 = -x_1 + 4x_2$ $\qquad x_1(0) = 2,\ x_2(0) = 1$;

8. $x_1' = 3x_1 - x_2$, $x_2' = x_1 + x_2$ $\qquad x_1(0) = 1,\ x_2(0) = 0$;

9. $y_1' = y_1 + y_3$, $y_2' = y_1 - y_2$, $y_3' = y_1 + y_3$

10. $y_1' = y_1 + 2y_3$, $y_2' = y_1 + y_3$, $y_3' = -y_1 + y_2 + 2y_3$

11. $x_1' = x_1 - x_3$, $x_2' = x_1 + x_2 + x_3$, $x_3' = 2x_1 + x_2$

Solutions

1. 1) general solution. **Eigenvalues:** $A=\begin{pmatrix}-2&4\\1&1\end{pmatrix}$, $|A-\lambda E|=\begin{vmatrix}-2-\lambda&4\\1&1-\lambda\end{vmatrix}=\lambda^2+\lambda-6=0$ yields $\lambda=-3,2$.

$\lambda=2$: $\begin{pmatrix}-4&4\\1&-1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $v_1-v_2=0$, choose $v_2=1$, then $v_1=1$, $\vec{y}_a(x)=\begin{pmatrix}1\\1\end{pmatrix}e^{2x}$.

$\lambda=-3$: $\begin{pmatrix}1&4\\1&4\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $v_1+4v_2=0$, choose $v_2=1$, then $v_1=-4$, $\vec{y}_b(x)=\begin{pmatrix}-4\\1\end{pmatrix}e^{-3x}$.

General solution $\vec{y}(x)=a\vec{y}_a+b\vec{y}_b=a\begin{pmatrix}e^{2x}\\e^{2x}\end{pmatrix}+b\begin{pmatrix}-4e^{-3x}\\e^{-3x}\end{pmatrix}=\begin{pmatrix}ae^{2x}-4be^{-3x}\\ae^{2x}+be^{-3x}\end{pmatrix}$.

Proper form: $y_1(x)=ae^{2x}-4be^{-3x}$, $y_2(x)=ae^{2x}+be^{-3x}$, $x\in I\!R$.

Remark: Fundamental matrix $Y(x)=\begin{pmatrix}e^{2x}&-4e^{-3x}\\e^{2x}&e^{-3x}\end{pmatrix}$.

For $x\sim\infty$ we get $y_1(x)\sim ae^{2x}$, $y_2(x)\sim ae^{2x}$.

The stationary solution $y_1(x)=y_2(x)=0$, or the equilibrium $(0,0)$ are unstable.

Bonus: $(0,0)$ is a saddle.

Elimination: From (#2) $y_1=y_2'-y_2$ $(*)$, into (#1) yields $y_2''+y_2'-6y_2=0$, char. num. $\lambda=-3,2$, solution $y_2(x)=ae^{2x}+be^{-3x}$, from $(*)$ we get $y_1(x)=ae^{2x}-4be^{-3x}$.

2) **Init. conditions** yield $y_1(x)=4e^{-3x}$, $y_2(x)=-e^{-3x}$, $x\in I\!R$.

2. 1) general solution. **Eigenvalues:** $A=\begin{pmatrix}2&1\\1&2\end{pmatrix}$, $\begin{vmatrix}2-\lambda&1\\1&2-\lambda\end{vmatrix}=\lambda^2-4\lambda+3=0$ yields $\lambda=1,3$.

$\lambda=1$: $\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $v_1+v_2=0$, choose $v_2=-1$, then $v_1=1$, $\vec{y}_a(x)=\begin{pmatrix}1\\-1\end{pmatrix}e^{x}$.

$\lambda=3$: $\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $v_1-v_2=0$, choose $v_2=1$, then $v_1=1$, $\vec{y}_b(x)=\begin{pmatrix}1\\1\end{pmatrix}e^{3x}$.

General solution $\vec{y}(x)=a\vec{y}_a+b\vec{y}_b=a\begin{pmatrix}e^{x}\\-e^{x}\end{pmatrix}+b\begin{pmatrix}e^{3x}\\e^{3x}\end{pmatrix}=\begin{pmatrix}ae^{x}+be^{3x}\\-ae^{x}+be^{3x}\end{pmatrix}$.

Proper form: $y_1(x)=ae^{x}+be^{3x}$, $y_2(x)=-ae^{x}+be^{3x}$, $x\in I\!R$.

Remark: Fundamental matrix $Y(x)=\begin{pmatrix}e^{x}&e^{3x}\\-e^{x}&e^{3x}\end{pmatrix}$.

For $x\sim\infty$ we get $y_1(x)\sim b\,e^{3x}$, $y_2(x)\sim b\,e^{3x}$.

The stationary solution $y_1(x)=y_2(x)=0$, or the equilibrium $(0,0)$ are unstable.

Bonus: $(0,0)$ is an unstable node (knot).

Elimination: From (#1) $y_2=y_1'-2y_1$ $(*)$, into (#2) yields $y_1''-4y_1'+3y_1=0$, char. num. $\lambda=1,3$, solution $y_1(x)=ae^{x}+be^{3x}$, from $(*)$ we get $y_2(x)=-ae^{x}+be^{3x}$.

2) **Init. conditions** yield $y_1(x)=e^{3x}+e^{x}$, $y_2(x)=e^{3x}-e^{x}$, $x\in I\!R$.

3. 1) general solution. **Eigenvalues:** $A=\begin{pmatrix}1&-3\\3&1\end{pmatrix}$, $\begin{vmatrix}1-\lambda&-3\\3&1-\lambda\end{vmatrix}=\lambda^2-2\lambda+10=0$ yields $\lambda=1\pm 3i$.

$\lambda=1-3i$: $\begin{pmatrix}3i&-3\\3&3i\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $3iv_1-3v_2=0$, choose $v_2=1$, then $v_1=-i$,

so $\vec{y}_C(x)=\begin{pmatrix}-i\\1\end{pmatrix}e^{(1-3i)x}=\begin{pmatrix}-i\\1\end{pmatrix}[e^x\cos(3x)-ie^x\sin(3x)]=\begin{pmatrix}-ie^x\cos(3x)-e^x\sin(3x)\\e^x\cos(3x)-ie^x\sin(3x)\end{pmatrix}$.

We take $\vec{y}_a(x)=\mathrm{Re}(\vec{y}_C)=\begin{pmatrix}-e^x\sin(3x)\\e^x\cos(3x)\end{pmatrix}$, $\vec{y}_b(x)=\mathrm{Im}(\vec{y}_C)=\begin{pmatrix}-e^x\cos(3x)\\-e^x\sin(3x)\end{pmatrix}$.

General solution

$$\vec{y}(x)=(-a)\vec{y}_a+(-b)\vec{y}_b=a\begin{pmatrix}e^x\sin(3x)\\-e^x\cos(3x)\end{pmatrix}+b\begin{pmatrix}e^x\cos(3x)\\e^x\sin(3x)\end{pmatrix}$$
$$=\begin{pmatrix}ae^x\sin(3x)+be^x\cos(3x)\\-ae^x\cos(3x)+be^x\sin(3x)\end{pmatrix}.$$

Proper form: $y_1(x)=ae^x\sin(3x)+be^x\cos(3x)$, $y_2(x)=-ae^x\cos(3x)+be^x\sin(3x)$, $x\in I\!R$.

Remark: Fundamental matrix $Y(x)=\begin{pmatrix}e^x\sin(3x)&e^x\cos(3x)\\-e^x\cos(3x)&e^x\sin(3x)\end{pmatrix}$.

For $x\sim\infty$ we cannot simplify the solution.

The stationary solution $y_1(x)=y_2(x)=0$, or the equilibrium $(0,0)$ are unstable.

Bonus: $(0,0)$ is an unstable focus.
Elimination: From (#1) $y_2=\frac{1}{3}y_1-\frac{1}{3}y_1'$ $(*)$, into (#2) yields $\frac{1}{3}y_1''-\frac{2}{3}y_1'+\frac{10}{3}y_1=0$, char. num. $\lambda=1\pm 3j$, solution $y_1(x)=ae^x\sin(3x)+be^x\cos(3x)$, from $(*)$ we get $y_2(x)=-ae^x\cos(3x)+be^x\sin(3x)$.
2) **Init. conditions** yield $y_1(x)=e^x[\cos(3x)-\sin(3x)]$, $y_2(x)=e^x[\cos(3x)+\sin(3x)]$, $x\in I\!R$.

4. 1) general solution. **Eigenvalues:** $A=\begin{pmatrix}2&-3\\3&-4\end{pmatrix}$, $\begin{vmatrix}2-\lambda&-3\\3&-4-\lambda\end{vmatrix}=(\lambda+1)^2=0$ yields $\lambda=-1$ $(2\times)$.
$\lambda=-1$: $\begin{pmatrix}3&-3\\3&-3\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $3v_1-3v_2=0$, choose $v_2=1$, then $v_1=1$, $\vec{y}_a(x)=\begin{pmatrix}1\\1\end{pmatrix}e^{-x}$.
Second solution: $\begin{pmatrix}3&-3\\3&-3\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$, tedy $3v_1-3v_2=1$, choice $v_2=0$ yields $v_1=\frac{1}{3}$,
$\vec{y}_b(x)=\left[\begin{pmatrix}1\\1\end{pmatrix}x+\begin{pmatrix}\frac{1}{3}\\0\end{pmatrix}\right]e^{-x}=\begin{pmatrix}(x+\frac{1}{3})e^{-x}\\x\,e^{-x}\end{pmatrix}$.
General solution $\vec{y}(x)=a\vec{y}_a+b\vec{y}_b=a\begin{pmatrix}e^{2x}\\e^{2x}\end{pmatrix}+b\begin{pmatrix}(x+\frac{1}{3})e^{-x}\\x\,e^{-x}\end{pmatrix}=\begin{pmatrix}ae^{-x}+b(x+\frac{1}{3})e^{-x}\\ae^{-x}+bx\,e^{-x}\end{pmatrix}$.
Proper form: $y_1(x)=ae^{-x}+b(x+\frac{1}{3})e^{-x}$, $y_2(x)=ae^{-x}+bx\,e^{-x}$, $x\in I\!R$.
For $x\sim\infty$ we get $y_1(x)\sim 3bx\,e^{2x}$, $y_2(x)\sim 3bx\,e^{2x}$.
Remark: If we use constant $3b$ in the combination, we get
$y_1(x)=ae^{-x}+b(3x+1)e^{-x}$, $y_2(x)=ae^{-x}+3bx\,e^{-x}$.
The stationary solution $y_1(x)=y_2(x)=0$, or the equilibrium $(0,0)$ are stable. We can see that $\vec{y}\to\vec{0}$, after all.
Bonus: $(0,0)$ is an stable node (knot).
Elimination: From (#1) $y_2=\frac{2}{3}y_1-\frac{1}{3}y_1'$ $(*)$, into (#2) yields $\frac{1}{3}y_2''+\frac{2}{3}y_2'+\frac{1}{3}y_2=0$, that is, $y_2''+2y_2'+y_2=0$, char. num. $\lambda=-1$ $(2\times)$, solution $y_1(x)=ae^{-x}+bx\,e^{-x}$, from $(*)$ we get $y_2(x)=ae^{-x}+b\big(x-\frac{1}{3}\big)e^{-x}$.
Remark: This general solution can be obtained from the one of matrix approach by choosing $a=\tilde{a}-\frac{1}{3}$.
Remark: Fundamental matrix $Y(x)=\begin{pmatrix}e^{-x}&3x\,e^{-x}\\e^{-x}&(3x-1)e^{-x}\end{pmatrix}=\begin{pmatrix}e^{-x}&(3x+1)e^{-x}\\e^{-x}&3x\,e^{-x}\end{pmatrix}$.
2) **Init. conditions** yield $y_1(x)=(3x+2)e^{-x}$, $y_2(x)=(3x+1)e^{-x}$, $x\in I\!R$.

5. 1) general solution. **Eigenvalues:** $A=\begin{pmatrix}1&4\\3&2\end{pmatrix}$, $\begin{vmatrix}1-\lambda&4\\3&2-\lambda\end{vmatrix}=\lambda^2-3\lambda-10=0$ yields $\lambda=-2,5$.
$\lambda=-2$: $\begin{pmatrix}3&4\\3&4\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $3v_1+4v_2=0$, choose $v_2=-3$, then $v_1=4$, $\vec{y}_a(x)=\begin{pmatrix}4\\-3\end{pmatrix}e^{-2x}$.
$\lambda=5$: $\begin{pmatrix}-4&4\\3&-3\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $v_1-v_2=0$, choose $v_2=1$, then $v_1=1$, $\vec{y}_b(x)=\begin{pmatrix}1\\1\end{pmatrix}e^{5x}$.
General solution $\vec{y}(x)=a\vec{y}_a+b\vec{y}_b=a\begin{pmatrix}4e^{-2x}\\-3e^{-2x}\end{pmatrix}+b\begin{pmatrix}e^{5x}\\e^{5x}\end{pmatrix}=\begin{pmatrix}4ae^{-2x}+be^{5x}\\-3ae^{-2x}+be^{5x}\end{pmatrix}$.
Proper form: $y_1(x)=4ae^{-2x}+be^{5x}$, $y_2(x)=-3ae^{-2x}+be^{5x}$, $x\in I\!R$.
Remark: Fundamental matrix $Y(x)=\begin{pmatrix}4e^{-2x}&e^{5x}\\-3e^{-2x}&e^{5x}\end{pmatrix}$.
For $x\sim\infty$ we get $y_1(x)\to 0$, $y_2(x)\to 0$.
The stationary solution $y_1(x)=y_2(x)=0$, or the equilibrium $(0,0)$ are unstable.
Bonus: $(0,0)$ is a saddle.
Elimination: From (#1) $y_2=\frac{1}{4}y_1'-\frac{1}{4}y_1$ $(*)$, into (#2) yields $\frac{1}{4}y_1''-\frac{3}{4}y_1'-\frac{10}{4}y_1=0$, that is, $y_1''-3y_1'-10y_1=0$, char. num. $\lambda=-2,5$, solution $y_1(x)=ae^{-2x}+be^{5x}$, from $(*)$ we get $y_2(x)=-\frac{3}{4}ae^{-2x}+be^{5x}$. If we replace a with $4a$, we get the same solution as we obtained using matrices.
2) **Init. conditions** yield $y_1(x)=4e^{-2x}-e^{5x}$, $y_2(x)=-3e^{-2x}-e^{5x}$, $x\in I\!R$.

6. 1) general solution. **Eigenvalues:** $A=\begin{pmatrix}1&-1\\2&-1\end{pmatrix}$, $\begin{vmatrix}1-\lambda&-1\\2&-1-\lambda\end{vmatrix}=\lambda^2+1=0$ yields $\lambda=\pm i$.
$\lambda=i$: $\begin{pmatrix}1-i&-1\\2&-1-i\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $(1-i)v_1-v_2=0$, choose $v_1=1$, then $v_2=1-i$,
so $\vec{x}_C(t)=\begin{pmatrix}1\\1-i\end{pmatrix}e^{it}=\begin{pmatrix}1\\1-i\end{pmatrix}[\cos(t)+i\sin(t)]=\begin{pmatrix}\cos(t)+i\sin(t)\\\cos(t)+\sin(t)+i[\sin(t)-\cos(t)]\end{pmatrix}$.

We take $\vec{x}_a(t)=\operatorname{Im}(\vec{x}_C)=\begin{pmatrix}\sin(t)\\ \sin(t)-\cos(t)\end{pmatrix}$, $\vec{x}_b(t)=\operatorname{Re}(\vec{x}_C)=\begin{pmatrix}\cos(t)\\ \sin(t)+\cos(t)\end{pmatrix}$.
General solution
$$\vec{x}(t)=a\vec{x}_a+b\vec{x}_b=a\begin{pmatrix}\sin(t)\\ \sin(t)-\cos(t)\end{pmatrix}+b\begin{pmatrix}\cos(t)\\ \sin(t)+\cos(t)\end{pmatrix}$$
$$=\begin{pmatrix}a\sin(t)+b\cos(t)\\ a[\sin(t)-\cos(t)]+b[\sin(t)+\cos(t)]\end{pmatrix}.$$
Proper form: $x_1(t)=a\sin(t)+b\cos(t)$, $x_2(t)=a[\sin(t)-\cos(t)]+b[\sin(t)+\cos(t)]$, $t\in I\!R$.
Remark: Fundamental matrix $X(t)=\begin{pmatrix}\sin(t) & \cos(t)\\ \sin(t)-\cos(t) & \sin(t)+\cos(t)\end{pmatrix}$.
For $t\sim\infty$ we cannot simplify the solution. It is bounded.
The stationary solution $y_1(x)=y_2(x)=0$, or the equilibrium $(0,0)$ are unstable.
Bonus: $(0,0)$ is a center.
Elimination: From (#1) $x_2=x_1-\dot{x}_1$ $(*)$, into (#2) yields $\ddot{x}_1+x_1=0$, char. num. $\lambda=\pm j$, solution $x_1(t)=a\sin(t)+b\cos(t)$, from $(*)$ we get $x_2(t)=a[\sin(t)-\cos(t)]+b[\sin(t)+\cos(t)]$.
2) **Init. conditions** yield $x_1(t)=\sin(t)+\cos(t)$, $x_2(t)=2\sin(t)$, $t\in I\!R$.

7. 1) general solution. **Eigenvalues:** $A=\begin{pmatrix}2 & 1\\ -1 & 4\end{pmatrix}$, $\begin{vmatrix}2-\lambda & 1\\ -1 & 4-\lambda\end{vmatrix}=\lambda^2-6\lambda+9=0$ yields $\lambda=3$ $(2\times)$.
$\lambda=3$: $\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$, $v_1-v_2=0$, choose $v_2=1$, then $v_1=1$, $\vec{x}_a(t)=\begin{pmatrix}1\\ 1\end{pmatrix}e^{3t}$.
Second solution: $\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$, so $-v_1+v_2=1$, choice $v_2=1$ yields $v_1=0$,
$\vec{x}_b(t)=\left[\begin{pmatrix}1\\ 1\end{pmatrix}t+\begin{pmatrix}0\\ 1\end{pmatrix}\right]e^{3t}=\begin{pmatrix}t\,e^{3t}\\ (t+1)e^{3t}\end{pmatrix}$.
General solution $\vec{x}(t)=a\vec{x}_a+b\vec{x}_b=a\begin{pmatrix}e^{3t}\\ e^{3t}\end{pmatrix}+b\begin{pmatrix}t\,e^{3t}\\ (t+1)e^{3t}\end{pmatrix}=\begin{pmatrix}ae^{3t}+bt\,e^{3t}\\ ae^{3t}+b(t+1)e^{3t}\end{pmatrix}$.
Proper form: $x_1(t)=ae^{3t}+bt\,e^{3t}$, $x_2(t)=ae^{3t}+b(t+1)e^{3t}$, $t\in I\!R$.
Remark: Fundamental matrix $X(t)=\begin{pmatrix}e^{3t} & t\,e^{3t}\\ e^{3t} & (t+1)e^{3t}\end{pmatrix}$.
For $t\sim\infty$ we get $x_1(t)\sim bt\,e^{3t}$, $x_2(t)\sim bt\,e^{3t}$.
The stationary solution $y_1(x)=y_2(x)=0$, or the equilibrium $(0,0)$ are unstable.
Bonus: $(0,0)$ is an unstable node (knot).
Elimination: From (#1) $x_2=\dot{x}_1-2x_1$ $(*)$, into (#2) yields $\ddot{x}_1-6\dot{x}_1+9x_1=0$, char. num. $\lambda=3$ $(2\times)$, solution $x_1(t)=ae^{3t}+bt\,e^{3t}$, from $(*)$ we get $x_2(t)=ae^{3t}+b(t+1)e^{3t}$.
2) **Init. conditions** yield $x_1(t)=(2-t)e^{3t}$, $x_2(t)=(1-t)e^{3t}$, $t\in I\!R$.

8. 1) general solution. **Eigenvalues:** $A=\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}$, $\begin{vmatrix}3-\lambda & -1\\ 1 & 1-\lambda\end{vmatrix}=\lambda^2-4\lambda+4=0$ yields $\lambda=2$ $(2\times)$.
$\lambda=2$: $\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$, $v_1-v_2=0$, choose $v_2=1$, then $v_1=1$, $\vec{x}_a(t)=\begin{pmatrix}1\\ 1\end{pmatrix}e^{2t}$.
Second solution: $\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$, so $v_1-v_2=1$, choice $v_1=0$ yields $v_2=-1$,
$\vec{x}_b(t)=\left[\begin{pmatrix}1\\ 1\end{pmatrix}t+\begin{pmatrix}0\\ -1\end{pmatrix}\right]e^{2t}=\begin{pmatrix}t\,e^{2t}\\ (t-1)e^{2t}\end{pmatrix}$.
General solution $\vec{x}(t)=a\vec{x}_a+b\vec{x}_b=a\begin{pmatrix}e^{2t}\\ e^{2t}\end{pmatrix}+b\begin{pmatrix}t\,e^{2t}\\ (t-1)e^{2t}\end{pmatrix}=\begin{pmatrix}ae^{2t}+bt\,e^{2t}\\ ae^{2t}+b(t-1)e^{2t}\end{pmatrix}$.
Proper form: $x_1(t)=ae^{2t}+bt\,e^{2t}$, $x_2(t)=ae^{2t}+b(t-1)e^{2t}$, $t\in I\!R$.
Remark: Fundamental matrix $X(t)=\begin{pmatrix}e^{2t} & t\,e^{2t}\\ e^{2t} & (t-1)e^{2t}\end{pmatrix}$.
For $t\sim\infty$ we get $x_1(t)\sim bt\,e^{2t}$, $x_2(t)\sim bt\,e^{2t}$.
The stationary solution $y_1(x)=y_2(x)=0$, or the equilibrium $(0,0)$ are unstable.
Bonus: $(0,0)$ is an unstable node (knot).
Elimination: From (#1) $x_2=3x_1-x_1'$ $(*)$, into (#2) yields $x_1''-4x_1'+4x_1=0$, char. num. $\lambda=2$ $(2\times)$, solution $x_1(t)=ae^{2t}+bt\,e^{2t}$, from $(*)$ we get $x_2(t)=ae^{2t}+b(t-1)e^{2t}$.

2) **Init. conditions** yield $x_1(t)=(t+1)e^{2t}$, $x_2(t)=t\,e^{2t}$, $t\in I\!R$.

9. Eigenvalues: $A=\begin{pmatrix}1&0&1\\1&-1&0\\1&0&1\end{pmatrix}$, $\begin{vmatrix}1-\lambda&0&1\\1&-1-\lambda&0\\1&0&1-\lambda\end{vmatrix}=$
$=-\lambda^3+\lambda^2+2\lambda=-\lambda(\lambda^2-\lambda-2)=0$ yields $\lambda=0,-1,2$.

$\lambda=0$: $\begin{pmatrix}1&0&1\\1&-1&0\\1&0&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, reduction $\begin{pmatrix}1&0&1\\1&-1&0\\1&0&1\end{pmatrix}\sim\begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}$, choice $v_3=-1$ yields $v_2=1$, $v_1=1$, $\vec{y}_a(x)=\begin{pmatrix}1\\1\\-1\end{pmatrix}$.

$\lambda=-1$: $\begin{pmatrix}2&0&1\\1&0&0\\1&0&2\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, reduction $\begin{pmatrix}2&0&1\\1&0&0\\1&0&2\end{pmatrix}\sim\begin{pmatrix}1&0&0\\0&0&1\\0&0&0\end{pmatrix}$, choice $v_2=1$, then $v_1=0$, $v_3=0$, $\vec{y}_b(x)=\begin{pmatrix}0\\1\\0\end{pmatrix}e^{-x}$.

$\lambda=2$: $\begin{pmatrix}-1&0&1\\1&-3&0\\1&0&-1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, reduction $\begin{pmatrix}-1&0&1\\1&-3&0\\1&0&-1\end{pmatrix}\sim\begin{pmatrix}-1&0&1\\0&3&-1\\0&0&0\end{pmatrix}$, choice $v_3=3$ yields $v_2=1$, $v_1=3$, $\vec{y}_c(x)=\begin{pmatrix}3\\1\\3\end{pmatrix}e^{2x}$.

General solution $\vec{y}(x)=a\begin{pmatrix}1\\1\\-1\end{pmatrix}+b\begin{pmatrix}0\\e^{-x}\\0\end{pmatrix}+c\begin{pmatrix}3e^{2x}\\e^{2x}\\3e^{2x}\end{pmatrix}=\begin{pmatrix}a+3ce^{2x}\\a+be^{-x}+ce^{2x}\\-a+3ce^{2x}\end{pmatrix}$.

Proper form: $y_1(x)=a+3ce^{2x}$, $y_2(x)=a+be^{-x}+ce^{2x}$, $y_3(x)=-a+3ce^{2x}$, $x\in I\!R$.

Remark: Fundamental matrix $Y(x)=\begin{pmatrix}1&0&3e^{2x}\\1&e^{-x}&e^{2x}\\-1&0&3e^{2x}\end{pmatrix}$.

The stationary solution $y_1(x)=y_2(x)=0$, or the equilibrium $(0,0)$ are unstable.
Bonus: $(0,0)$ is a saddle.

Elimination: From (#2) $y_1=y_2'+y_2$ $(*)$, into (#1) a (#3) yields $\left\{\begin{matrix}(1^*)\ y_2''=y_2+y_3\\(2^*)\ y_3'=y_2'+y_2+y_3\end{matrix}\right\}$, z (#1*) $y_3=y_2''-y_2$ $(\star)$, into (#2*) yields $y_2'''-y_2''-2y_2'=0$. Char. num. $\lambda=0,-1,2$, solution $y_2(x)=a+be^{-x}+ce^{2x}$, from $(\star)$ we get $y_3(x)=-a+3ce^{2x}$, from $(*)$ we get $y_1(x)=a+3ce^{2x}$.

10. Eigenvalues: $A=\begin{pmatrix}1&0&2\\1&0&1\\-1&1&2\end{pmatrix}$, $\begin{vmatrix}1-\lambda&0&2\\1&-\lambda&1\\-1&1&2-\lambda\end{vmatrix}=-\lambda^3+3\lambda^2-3\lambda+1=$
$=-(\lambda-1)^3=0$ yields $\lambda=1$ $(3\times)$.

$\lambda=1$: $\begin{pmatrix}0&0&2\\1&-1&1\\-1&1&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, reduction $\begin{pmatrix}0&0&2\\1&-1&1\\-1&1&1\end{pmatrix}\sim\begin{pmatrix}1&-1&0\\0&0&1\\0&0&0\end{pmatrix}$, choice $v_2=1$ yields $v_1=1$, $v_3=0$, $\vec{y}_a(x)=\begin{pmatrix}1\\1\\0\end{pmatrix}e^x=\begin{pmatrix}e^x\\e^x\\0\end{pmatrix}$.

Second solution: $\begin{pmatrix}0&0&2\\1&-1&1\\-1&1&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$, reduction $\begin{pmatrix}0&0&2&1\\1&-1&1&1\\-1&1&1&0\end{pmatrix}\sim\begin{pmatrix}1&-1&0&\frac{1}{2}\\0&0&1&\frac{1}{2}\\0&0&0&0\end{pmatrix}$, choice $v_2=0$ yields $v_1=\frac{1}{2}$, $v_3=\frac{1}{2}$, $\vec{y}_b(x)=\left[\begin{pmatrix}1\\1\\0\end{pmatrix}x+\begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}\right]e^x=\begin{pmatrix}(x+\frac{1}{2})e^x\\x\,e^x\\\frac{1}{2}e^x\end{pmatrix}$.

Third solution: $\begin{pmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$, reduction $\begin{pmatrix} 0 & 0 & 2 & \frac{1}{2} \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$,
choice $v_2 = 0$ yields $v_1 = -\frac{1}{4}$, $v_3 = \frac{1}{4}$,

$$\vec{y}_c(x) = \left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x^2 + \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} x + \begin{pmatrix} -\frac{1}{4} \\ 0 \\ \frac{1}{4} \end{pmatrix}\right] e^x = \begin{pmatrix} (\frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{4})e^x \\ (\frac{1}{4}x^2)e^x \\ (\frac{1}{2}x + \frac{1}{4})e^x \end{pmatrix}.$$

General solution $\vec{y}(x) = a\begin{pmatrix} e^x \\ e^x \\ 0 \end{pmatrix} + (2b)\begin{pmatrix} (x+\frac{1}{2})e^x \\ x\,e^x \\ \frac{1}{2}e^x \end{pmatrix} + (4c)\begin{pmatrix} (\frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{4})e^x \\ (\frac{1}{4}x^2)e^x \\ (\frac{1}{2}x + \frac{1}{4})e^x \end{pmatrix}$

$$= a\begin{pmatrix} e^x \\ e^x \\ 0 \end{pmatrix} + b\begin{pmatrix} (2x+1)e^x \\ 2x\,e^x \\ e^x \end{pmatrix} + c\begin{pmatrix} (x^2+2x-1)e^x \\ x^2e^x \\ (2x+1)e^x \end{pmatrix}$$

$$= \begin{pmatrix} ae^x + b(2x+1)e^x + c(x^2+2x-1)e^x \\ ae^x + 2bx\,e^x + cx^2e^x \\ be^x + c(2x+1)e^x \end{pmatrix}.$$

Proper form: $y_1(x) = ae^x + b(2x+1)e^x + c(x^2+2x-1)e^x$, $y_2(x) = ae^x + 2bx\,e^x + cx^2e^x$, $y_3(x) = be^x + c(2x+1)e^x$, $x \in I\!R$.

Remark: Fundamental matrix $Y(x) = \begin{pmatrix} e^x & (2x+1)e^x & (x^2+2x-1)e^x \\ e^x & 2x\,e^x & x^2e^x \\ 0 & e^x & (2x+1)e^x \end{pmatrix}$.

The stationary solution $y_1(x) = y_2(x) = 0$, or the equilibrium $(0,0)$ are unstable.
Bonus: $(0,0)$ is an unstable node (knot).

Elimination: From (#2) $y_3 = y_2' - y_1$ $(*)$, into (#1) a (#3) yields $\left\{\begin{matrix} (1^*)\ 2y_2' - y_1' = y_1 \\ (2^*)\ y_2'' - 2y_2' - y_1' = y_2 - 3y_1 \end{matrix}\right\}$.
We cannot isolate y_1 nor y_3 from equations, so we first eliminate one of the derivatives. We try to get rid of y_1' using (#2*)−(#1*): $y_2'' - 4y_2' = y_2 - 4y_1$, so $y_1 = -\frac{1}{4}y_2'' + y_2' + \frac{1}{4}y_2$ $(\star)$, into (#1*) yields $\frac{1}{4}y_2''' - \frac{3}{4}y_2'' + \frac{3}{4}y_2' - \frac{1}{4}y_2 = 0$, that is, $y_2''' - 3y_2'' + 3y_2' - y_2 = 0$. Char. num. $\lambda = 1$ $(3\times)$, solution $y_2(x) = ae^x + bx\,e^x + cx^2e^x$, from $(\star)$ we get $y_1(x) = ae^x + b\big(x+\frac{1}{2}\big)e^x + c\big(x^2 + x - \frac{1}{2}\big)e^x$, from $(*)$ we have $y_3(x) = b\frac{1}{2}e^x + c\big(x + \frac{1}{2}\big)e^x$.

11. Eigenvalues: $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$, $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 1-\lambda & 1 \\ 2 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 - 2\lambda$
$= -\lambda(\lambda^2 - 2\lambda + 2) = 0$ yields $\lambda = 0, 1 \pm i$.

$\lambda = 0$: $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, reduction $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, choice $v_3 = 1$ yields $v_2 = -2$, $v_1 = 1$, $\vec{x}_a(t) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

$\lambda = 1 - i$: $\begin{pmatrix} i & 0 & -1 \\ 1 & i & 1 \\ 2 & 1 & i-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, reduction $\begin{pmatrix} i & 0 & -1 \\ 1 & i & 1 \\ 2 & 1 & i-1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & -1-i \\ 0 & 0 & 0 \end{pmatrix}$,
choice $v_3 = 1$, $v_1 = -i$, $v_2 = 1 + i$,

so $\vec{x}_C(t) = \begin{pmatrix} -i \\ 1+i \\ 1 \end{pmatrix} e^{(1-i)t} = \begin{pmatrix} -i \\ 1+i \\ 1 \end{pmatrix} e^t[\cos(t) - i\sin(t)]$

$$= \begin{pmatrix} -e^t\sin(t) - ie^t\cos(t) \\ e^t[\cos(t) + \sin(t)] + ie^t[-\sin(t) + \cos(t)] \\ e^t\cos(t) - ie^t\sin(t) \end{pmatrix}.$$

We take $\vec{x}_b(t) = \mathrm{Re}(\vec{x}_C) = \begin{pmatrix} -e^t\sin(t) \\ e^t[\sin(t) + \cos(t)] \\ e^t\cos(t) \end{pmatrix}$, $\vec{x}_c(t) = \mathrm{Im}(\vec{x}_C) = \begin{pmatrix} -e^t\cos(t) \\ e^t[-\sin(t) + \cos(t)] \\ -e^t\sin(t) \end{pmatrix}$,

General solution:

$$\vec{x}(t) = a\begin{pmatrix}1\\-2\\1\end{pmatrix} + b\begin{pmatrix}-e^t\sin(t)\\-e^t[\sin(t)+\cos(t)]\\e^t\cos(t)\end{pmatrix} + c\begin{pmatrix}-e^t\cos(t)\\e^t[-\sin(t)+\cos(t)]\\-e^t\sin(t)\end{pmatrix}$$
$$= \begin{pmatrix}a-be^t\sin(t)-ce^t\cos(t)\\-2a+be^t\sin(t)+be^t\cos(t)-ce^t\sin(t)+ce^t\cos(t)\\a+be^t\cos(t)-ce^t\sin(t)\end{pmatrix}.$$

Proper form: $x_1(t)=a-be^t\sin(t)-ce^t\cos(t)$, $x_2(t)=-2a+be^t\sin(t)+be^t\cos(t)-ce^t\sin(t)+ce^t\cos(t)$, $x_3(t)=a+be^t\cos(t)-ce^t\sin(t)$, $t\in I\!R$.

Remark: Fundamental matrix $X(t)=\begin{pmatrix}1 & e^t\sin(t) & e^t\cos(t)\\-2 & -e^t[\sin(t)+\cos(t)] & e^t[\sin(t)-\cos(t)]\\1 & -e^t\cos(t) & e^t\sin(t)\end{pmatrix}$.

The stationary solution $y_1(x)=y_2(x)=0$, or the equilibrium $(0,0)$ are unstable.
Bonus: $(0,0)$ is an unstable focus.

Elimination: From (#1) $x_3=x_1-x_1'$ $(*)$, into (#2) a (#3) yields $\left\{\begin{matrix}(1^*)\ x_1'+x_2'=2x_1+x_2\\(2^*)\ x_1'-x_1''=2x_1+x_2\end{matrix}\right\}$,
from (#2*) $x_2=x_1'-x_1''-2x_1$ $(\star)$, into (#1*) yields $x_1'''-2x_1''+2x_1'=0$. Char. num. $\lambda=0,1\pm j$, solution $x_1(t)=a+be^t\sin(t)+ce^t\cos(t)$,
from $(\star)$ we get $x_2(t)=-2a-be^t\sin(t)-be^t\cos(t)+ce^t\sin(t)-ce^t\cos(t)$,
from $(*)$ we get $x_3(t)=a-be^t\cos(t)+ce^t\sin(t)$.
Remark: We see that elimination yielded two vectors of the base with opposite signs compared to matrix approach, but they generate the same space, of course.