DEN: lab #8 (coronaversion)

All calculations can be done in one's head or by hand, just like at the exam. You can use calculators, Maple or some other computing device if it is really necesary.

First we practice the basic two root-finding methods.

1. Consider the function $f(x) = x^2 - 3x + 1$. We are looking for its root.

   a) Apply the bisection method to this problem on the interval $[1, 5]$, show two steps of iteration. Comment on your steps, above all on your decision making, so that the examiner can see that you know what you are doing. (A novice should be able to understand how the bisection method works based on your comments.)

   b) Apply the Newton method to this problem with the initial guess $x_0 = 1$. Find the first three approximations (that is, do two steps of iteration).

2. Consider the equation $x - 3 = \dfrac{1}{x^2}$. We are looking for its solution.
   Rewrite it as a root-finding problem.

   a) Apply the bisection method to this problem on the interval $[1, 5]$, show two steps of iteration. Comment on your steps, above all on your decision making.

   b) Apply the Newton method to this problem with the initial guess $x_0 = 1$. Find the first three approximations (that is, do two steps of iteration).
   Start by preparing a specific iterative formula for x_{k+1} by substituting your f into the general formula (try to simplify), then just substitute into it.

3. Apply the Newton method to the problem of finding the root of the function $f(x) = x^2 - x + 1$, with the initial guess $x_0 = 2$. Find the first five approximations (that is, do four iteration steps).
   What do you think about this situation? Based on what you learned in the lecture, can you make a guess regarding the situation?

4. Review the basic three stopping conditions from the lecture.
   Consider the function $\ln(x) = 4 - \sqrt{x}$. Its root is near number five, and the function is a bit flat around this root. Assume that we started some iterative method that generates sequences x_k converging to this root. We actually started it three times in parallel, each with a different stopping condition.

   a) One of the runs will have a tendency to stop prematurely, that is, its stopping conditon is likely to be satisfied before the distance of x_k from the solution drops below the given tolerance. Can you explain which condition it is and why?

   b) Based on what we know about the function, we cannot make a guess about the order in which the three runs stop. However, there is a pair of conditions where we do have an indication in which order they will become true. Can you find such a pair and explain why it is so?

   c) Assume that a certain run stopped and offered an approximation $\hat{r}$ of the root. Do you know some procedure that could guarantee that the distance between this approximation and the actual root is smaller than the given ε? Explain.

5. Review from the lecture the notion of order of convergence of a method.
   The corresponding formula $|E_{k+1}| \approx c|E_k|^p$ has the disadvantage that even with the two errors being known, there are still two unknowns left. We therefore need to know one more iteration x_{k+2} with its error, then we can set up a second equality $|E_{k+2}| \approx c|E_{k+1}|^p$.
   Assume optimistically that these two relations are actually precise equations, and use them to derive a formula that would use three known errors E_k, E_{k+1}, E_{k+2} to estimate the order of method p.
   Remark: This formula will be used in the semestral project.