DEN: solutions for lab #8 (coronaversion)

1. a) $[a_0, b_0] = [1, 5]$. $f(1) = -1 < 0$, $f(5) = 11 > 0$, opposite signs.
Middle $m_0 = 3$, $f(3) = 1 > 0$, opposite sign versus $f(1)$, hence $[a_1, b_1] = [1, 3]$.
Middle $m_1 = 2$, $f(2) = -1 < 0$, opposite sign versus $f(3)$, hence $[a_2, b_2] = [2, 3]$.
Output $m_2 = 2.5$.
Remark: This level of comments would be enough for me on a test. If you write more, it is not wrong, but it will cost you time (and time is money).

b) Formula: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, $f'(x) = 2x - 3$.
$x_0 = 1$, $x_1 = 1 - \frac{f(1)}{f'(1)} = 0$, $x_2 = 0 - \frac{f(0)}{f'(0)} = \frac{1}{3}$.

2. Rewrite as equation: $x - 3 - \dfrac{1}{x^2} = 0$, so we take $f(x) = x - 3 - \dfrac{1}{x^2}$.

a) $[a_0, b_0] = [1, 5]$. $f(1) = -4 < 0$, $f(5) = 2 - \frac{1}{25} > 0$, opposite signs.
Middle $m_0 = 3$, $f(3) = -\frac{1}{9} < 0$, opposite sign versus $f(5)$, hence $[a_1, b_1] = [3, 5]$.
Middle $m_1 = 4$, $f(4) = 1 - \frac{1}{16} > 0$, opposite sign versus $f(3)$, hence $[a_2, b_2] = [3, 4]$.
Output $m_2 = 3.5$.

b) Formula:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k - 3 - \frac{1}{x_k^2}}{1 + \frac{2}{x_k^3}} = x_k - \frac{x_k^4 - 3x_k^3 - x_k}{x_k^3 + 2} = 3\frac{x_k^3 + x_k}{x_k^3 + 2}.$$

$x_0 = 1$, $x_1 = 3\frac{1^3+1}{1^3+2} = 2$, $x_2 = 3\frac{2^3+2}{2^3+2} = 3$.

Some students succumbed to the tempation of writing the equation as $x^3 - 3x^2 - 1 = 0$, then $f(x) = x^3 - 3x^2 - 1$. Gods punished them for it, and they had to work with a less user friendly formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - 3x_k^2 - 1}{3x_k^2 - 6x_k} = \frac{2x_k^3 - 3x_k^2 + 1}{3x_k^2 - 6x_k}.$$

Moreover, they encountered an interesting situation when $x_0 = 1$ leads to $x_1 = 0$, and the attempt to determine x_2 fails with division by zero problem.
Actually, this other solution is also correct.

3. Those who felt like it prepared the formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - x_k + 1}{2x_k - 1} = \frac{x_k^2 - 1}{2x_k - 1}.$$

$x_0 = 2$, $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$, $x_5 = 1$.
Iterations oscillate. It seems that the function is shaped like a valley that does not drop below the x-axis, and the iterations swing left and right in this valley. Since the given function is a polynomial of order two, we guess that it is a parabola above the x-axis, without any root.

4. a) Since the function is flat, there are places where the function has values smaller than the given $\varepsilon > 0$, but we are still further than ε from the root. If the algoritmus jumps into this place, it will stop prematurely.
However, if the algorithm is lucky, it will jump across this zone. That is, first both the function and the error of approximation are too large, and then both are small enough. Therefore, the value stopping condition is prone to stop prematurely, but it need not always happen.

b) As the numbers produced by the algorithm approach the root, they are close to five, and we have the relationship (rel. difference)$\approx\frac{1}{5}$(abs. difference).
So the relative difference is significantly smaller than the absolute difference, and thus the former should typically drop below the given ε before the latter does. Again, it may happen that these two conditions trigger simultaneously, but the relative will never be later than the absolute condition with this root.

c) We calculate $f(\hat{r}-\varepsilon)$, $f(\hat{r})$, $f(\hat{r}+\varepsilon)$. If we find opposite signs among these three numbers, then there must be a root within the range $(\hat{r}-\varepsilon, \hat{r}+\varepsilon)$, and therefore the error of $\hat{r}$ is smaller than ε.

5. Equations: $|E_{k+1}| = c|E_k|^p$ a $|E_{k+2}| = c|E_{k+1}|^p$.
We want to get rid of c, and this can be done by expressing c from one equation and substitutiing into the other. Or we can divide the first equation by the second one:

$$\frac{|E_{k+1}|}{|E_{k+2}|} = \frac{|E_k|^p}{|E_{k+1}|^p} \implies \frac{|E_{k+1}|}{|E_{k+2}|} = \Big(\frac{|E_k|}{|E_{k+1}|}\Big)^p.$$

We apply the logarithm to each side:

$$\ln\Big(\frac{|E_{k+1}|}{|E_{k+2}|}\Big) = p\ln\Big(\frac{|E_k|}{|E_{k+1}|}\Big) \implies p = \frac{\ln\Big(\frac{|E_{k+1}|}{|E_{k+2}|}\Big)}{\ln\Big(\frac{|E_k|}{|E_{k+1}|}\Big)} = \frac{\ln|E_{k+1}| - \ln|E_{k+2}|}{\ln|E_k| - \ln|E_{k+1}|}.$$

If somebody arrived at

$$p = \frac{\ln\Big(\frac{|E_{k+2}|}{|E_{k+1}|}\Big)}{\ln\Big(\frac{|E_{k+1}|}{|E_k|}\Big)} = \frac{\ln|E_{k+2}| - \ln|E_{k+1}|}{\ln|E_{k+1}| - \ln|E_k|},$$

then this is also correct.
By the way, the version with ratios of errors is better than the one with differences of logarithms, although it looks less attractive. Why? Because when the errors are really small, then the numbers $\ln(E_i)$ are really huge, moreover, they will differ by more orders than the errors themselves, which is just an invitation for numerical troubles. Ratios of errors are usually more computer-calculations friendly.