**DEN: solutions for lab #9 (coronaversion)**

To see how the iterations actually work we quote practical experiments below. For fair comparison, all of them use the absolute difference stopping condition with $\varepsilon = 0.0001$.

**1.** a) Iteration is $x_{k+1} = \varphi(x_k)$, hence

$$x_{k+1} = x_k^2 - 2x_k + 1.$$

Therefore $x_0 = 1$ (given), $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$, ...

b) $\varphi'(x) = 2x - 2$, hence $|\varphi'(1)| = 0$. This looks hopeful, but as we see above, the iteration enters a cycle.

What is happening here? The good feeling about $\varphi$ applies only to a neighborhood of $x_0$, but the iteration moves elsewhere where the derivative $\varphi'$ is markedly larger than 1. In fact, local evaluation of $\varphi'$ is only of limited usefulness.

Properly we should evaluate the derivative globally over an interval $I$ such that $\varphi$ maps it into itself, then the Banach contraction theorem applies. However, this is not always easy, in particular determining $I$ may be troublesome. We can simply ask whether $\varphi'$ is significantly smaller than 1 globally (everywhere), and if it is, then we can expect good convergence. However, it is more likely that we will not succeed because the derivative will be large somewhere. However, then we again do not really know anything, because our iteration may never go to those places with large derivative. A fairly hopeful situation is when we have a rough idea about the location of the root. Then we can start our iteration there and also look at $\varphi'$ locally there, and if the derivative is markedly less than 1, then there is a fairly good chance that our iteration will work out fine.

c,d) There are quite a few transformations posssible.

• $x^2 - 3x + 1 = 0$ means $3x = x^2 + 1$, hence $x = \frac{1}{3}(x^2 + 1)$.
We have $\varphi(x) = \frac{1}{3}(x^2 + 1)$, the iteration is $x_{k+1} = \frac{1}{3}(x_k^2 + 1)$.
Approximations are $x_0 = 1$ (given), $x_1 = \frac{2}{3}$, $x_2 = \frac{13}{27}$.
(The thirteen was not planned, just a lucky coincidence.)
$\varphi'(x) = \frac{2}{3}x$, hence $|\varphi'(1)| = \frac{2}{3}$. This number is somewhat smaller than one, so things look hopeful.
By the way, this iteration converges to the root $\approx 0.38$ after $N = 8$ steps.

• $x^2 - 3x + 1 = 0$ means $x^2 = 3x - 1$, hence $x = \sqrt{3x - 1}$.
We have $\varphi(x) = \sqrt{3x - 1}$, the iteration is $x_{k+1} = \sqrt{3x_k - 1}$.
This is a bit risky, hopefully negative numbers will not appear during iteration.
Approximations are $x_0 = 1$ (given), $x_1 = \sqrt{2}$, $x_2 = \sqrt{3\sqrt{2} - 1}$.
$\varphi'(x) = \frac{3}{2\sqrt{3x-1}}$, hence $|\varphi'(1)| = \frac{3}{2\sqrt{2}} \approx 1.06$. We have a number larger than 1, this is not very hopeful. On the other hand, it is close to 1 and those $x_k$ then move elsewhere, perhaps it will be better there. I would definitely try relaxation here with good chances of success.
By the way, this iteration converges to the root $\approx 2.62$ after $N = 19$ steps.

• $x^2 - 3x + 1 = 0$ means $x^2 - 3x = -1$, so $x(x - 3) = -1$, hence for instance $x = \frac{-1}{x-3} = \frac{1}{3-x}$.
We have $\varphi(x) = \frac{1}{3-x}$, the iteration is $x_{k+1} = \frac{1}{3-x_k}$.
Hopefully we will not hit a division by zero situaton.
Approximations are $x_0 = 1$ (given), $x_1 = \frac{1}{2}$, $x_2 = \frac{2}{5}$.
$\varphi'(x) = \frac{1}{(3-x)^2}$, hence $|\varphi'(1)| = \frac{1}{4}$. This is significantly less than one, this looks quite hopeful.
By the way, this iteration converges to the root $\approx 0.38$ after $N = 6$ steps. It is really fast.

• $x^2 - 3x + 1 = 0$ means $x^2 - 3x = -1$, so $x(x - 3) = -1$, this means $x - 3 = -\frac{1}{x}$, hence $x = 3 - \frac{1}{x}$.
We have $\varphi(x) = 3 - \frac{1}{x}$, the iteration is $x_{k+1} = 3 - \frac{1}{x_k}$.
Hopefully we will not hit a division by zero situaton.
Approximations are $x_0 = 1$ (given), $x_1 = 2$, $x_2 = \frac{5}{2}$.
$\varphi'(x) = \frac{1}{x^2}$, hence $|\varphi'(1)| = 1$. This is exactly the borderline. As iterations move (hopefully) towards a root (which one?), the derivative is bound to change and we have no idea how. So no guessing here.

By the way, this iteration converges to the root $\approx 2.62$ after $N = 7$ steps. So in the end it was quite good.

• One a bit crazy to round this out.
$x^2 - 3x + 1 = 0$ means $x^2 - 2x + 1 = x$ (we have already been here), but then $(x-1)^2 = x$, so $x - 1 = \pm\sqrt{x}$, hence $x = 1 \pm \sqrt{x}$.
We try the one with plus: $\varphi(x) = 1 + \sqrt{x}$, the iteration is $x_{k+1} = 1 + \sqrt{x_k}$.
Approximations are $x_0 = 1$ (given), $x_1 = 2$, $x_2 = 1 + \sqrt{2}$.
$\varphi'(x) = \frac{1}{2\sqrt{x}}$, hence $|\varphi'(1)| = \frac{1}{2}$. This is markedly smaller than one, this looks quite hopeful.
By the way, this iteration converges to the root $\approx 2.62$ after $N = 10$ steps.
We try the one with minus: $\varphi(x) = 1 - \sqrt{x}$, the iteration is $x_{k+1} = 1 - \sqrt{x_k}$.
Approximations are $x_0 = 1$ (given), $x_1 = 0$, $x_2 = 1$. We are in a cycle again.
$\varphi(x) = -\frac{1}{2\sqrt{x}}$, hence $|\varphi'(1)| = \frac{1}{2}$. This is less than one and looks hopeful, but as we saw, it did not work out at the end.
However, if we start at $x_0 = 0.9$, we reach the root $\approx 0.38$ after lazy $N = 46$ steps. On the other hand, an attempt with $x_0 = 1.1$ leads to a negative number under the root. If we decide to ride this out, the iteration gets to the interesting number $\approx 0.381996 - 0.000024i$, where the real part is quite close to the actual root (but not close enough in the sense of the given tolerance).

That was all I could think of, but some students surprised with more transforms.
• $x^2 - 3x + 1 = 0$ means $x^2 - x + 1 = 2x$, so $\frac{1}{2}(x^2 - x + 1) = x$, hence $\varphi(x) = \frac{1}{2}(x^2 - x + 1)$, the iteration is $x_{k+1} = \frac{1}{2}(x_k^2 - x_k + 1)$.
Approximations are $x_0 = 1$ (given), $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{8}$.
$\varphi'(x) = \frac{1}{2}(2x - 1)$, hence $|\varphi'(1)| = \frac{1}{2}$. This is markedly smaller than one, this looks quite hopeful.
By the way, this iteration converges to the root $\approx 0.38$ after $N = 6$ steps, it is quite good.

• $x^2 - 3x + 1 = 0$ means $x^2 - 3x = -1$, so $x(x-3) = -1$, then $x - 3 = -\frac{1}{x}$, hence $x = 3 - \frac{1}{x}$.
We have $\varphi(x) = 3 - \frac{1}{x}$, the iteration is $x_{k+1} = 3 - \frac{1}{x_k}$.
Hopefully we will not hit a division by zero situaton.
Approximations are $x_0 = 1$ (given), $x_1 = 2$, $x_2 = \frac{5}{2}$.
$\varphi'(x) = \frac{1}{x^2}$, hence $|\varphi'(1)| = 1$. This is exactly the borderline. As iterations move (hopefully) towards a root (which one?), the derivative is bound to change and we have no idea how. So no guessing here.
By the way, this iteration converges to the root $\approx 2.62$ after $N = 7$ steps. So it is quite good after all.

**2.** Try to find your transforms below, hopefully I thought of yours as well.
• From the equation $x - 3 = \frac{1}{x^2}$ we naturally get $x = 3 + \frac{1}{x^2}$.
So $\varphi(x) = 3 + \frac{1}{x^2}$, the iteration is $x_{k+1} = 3 + \frac{1}{x_k^2}$.
Hopefully we will not hit a division by zero situaton.
Approximations are $x_0 = 1$ (given), $x_1 = 4$, $x_2 = \frac{49}{16}$.
$\varphi'(x) = -\frac{2}{x^3}$, hence $|\varphi'(1)| = 2$. This does not look good, we can only hope that the iteration moves elsewhere where things look better.
We can guess (for instance by looking at the graph) that the root is somewhere between three and four, so we try $|\varphi'(3)| = \frac{2}{27}$, this is a very small number. So if the iteration manages to come to this place, it has a chance. And indeed, we see that $x_1 = 4$ and $x_2 \approx 3.06$, this is starting to look hopeful.
By the way, this iteration converges to the root $\approx 3.10$ after $N = 6$ steps. So it is rather good in the end.
Bonus: Relaxation with $\lambda = \frac{2}{3}$ uses the iterative function

$$\varphi_{2/3}(x) = \frac{2}{3} \cdot \left(3 + \frac{1}{x^2}\right) + \left(1 - \frac{2}{3}\right)x,$$

so $x_{k+1} = \frac{2}{3} \cdot \left(3 + \frac{1}{x_k^2}\right) + \left(1 - \frac{2}{3}\right)x_k$.

Therefore $x_1 = \frac{8}{3} + \frac{1}{3} = 3$.
The optimal $\lambda$ for a neighborhood of $x_0 = 1$ satisfies $\varphi'_\lambda(1) = 0$. We have

$$\varphi'_\lambda(x) = -\lambda\tfrac{2}{x^3} + (1-\lambda),$$

we substitute $x = 1$, set it equal to zero and obtain $\lambda = \frac{1}{3}$.
However, this value probably is not the best for our iteration, because it quickly moves elsewhere. Experiments confirm this.
The optimal $\lambda$ for a neighborhood of $x_0 = 3$ is $\lambda = \frac{27}{29} \approx 0.93$ and works rather well.

• If we want to use ready tools, we can first create a root type equation $x - 3 - \frac{1}{x^2} = 0$ and then apply the standard transformation to get $2x - 3 - \frac{1}{x^2} = x$.
So $\varphi(x) = 2x - 3 - \frac{1}{x^2}$, the iteration is $x_{k+1} = 2x_k - 3 - \frac{1}{x_k^2}$.
Hopefully we will not hit a division by zero situaton.
Approximations are $x_0 = 1$ (given), $x_1 = -2$, $x_2 = -\frac{29}{4}$.
$\varphi'(x) = 2 + \frac{2}{x^3}$, hence $|\varphi'(1)| = 4$. This is significantly larger than one.
We try $|\varphi'(3)| = \frac{56}{27}$, this is still too much. It looks bad and it is, experiments show fast divergence.
Bonus: Relaxation with $\lambda = \frac{2}{3}$ uses the iterative function

$$\varphi_{2/3}(x) = \tfrac{2}{3} \cdot \big(2x - 3 - \tfrac{1}{x^2}\big) + \big(1 - \tfrac{2}{3}\big)x,$$

so $x_{k+1} = \frac{2}{3} \cdot \big(2x - 3 - \frac{1}{x_k^2}\big) + \big(1 - \frac{2}{3}\big)x_k$.
Therefore $x_1 = -\frac{4}{3} + \frac{1}{3} = -1$, then $x_2 = -\frac{13}{3}$.
The optimal $\lambda$ for a neighborhood of $x_0 = 1$ satisfies $\varphi'_\lambda(1) = 0$. We have

$$\varphi'_\lambda(x) = \lambda\big(2 + \tfrac{2}{x^3}\big) + (1-\lambda),$$

we substitute $x = 1$, set it equal to zero and obtain $\lambda = -\frac{1}{3}$. This actually saves the iteration, it converges to the root after $N = 21$ steps.
The optimal $\lambda$ for a neighborhood of $x_0 = 3$ is $\lambda = -\frac{27}{29}$ and it shortens the run to great $N = 5$ steps.

• From the equation $x - 3 = \frac{1}{x^2}$ we can naturally move also to $x^3 - 3x^2 = 1$. We can follow several paths from here, for instance $x^3 = 1 + 3x^2$ and hence $x = (1 + 3x^2)^{1/3}$.
So $\varphi(x) = (1 + 3x^2)^{1/3}$, the iteration is $x_{k+1} = (1 + 3x_k^2)^{1/3}$.
Approximations are $x_0 = 1$ (given), $x_1 = 4^{1/3}$.
$\varphi'(x) = \frac{2x}{(1+3x^2)^{2/3}}$, hence $|\varphi'(1)| = \frac{2}{4^{2/3}} = \frac{1}{2^{1/3}} \approx 0.79$. This is smaller than one, but not by much, so we are mildly optimistic.
By the way, experiments show convergence, after $N = 23$ steps we arrive at the root $\approx 3.10$.
Bonus: Relaxation with $\lambda = \frac{2}{3}$ uses the iterative function

$$\varphi_{2/3}(x) = \tfrac{2}{3} \cdot (1 + 3x^2)^{1/3} + \big(1 - \tfrac{2}{3}\big)x,$$

so $x_{k+1} = \frac{2}{3} \cdot (1 + 3x_k^2)^{1/3} + \big(1 - \frac{2}{3}\big)x_k$.
Therefore $x_1 = \frac{2}{3} \cdot 4^{1/3} + \frac{1}{3}$.
The optimal $\lambda$ for a neighborhood of $x_0 = 1$ satisfies $\varphi'_\lambda(1) = 0$. We have

$$\varphi'_\lambda(x) = \lambda\tfrac{2x}{(1+3x^2)^{2/3}} + (1-\lambda),$$

we substitute $x = 1$, set it equal to zero and obtain $\lambda = \frac{2^{1/3}}{2^{1/3}-1} \approx 4.85$. This is a fairly strong relaxation. Experiments show convergence after $N = 20$ steps, a modest improvement.
It is of more interest to ptimize aroun three, we get $\lambda = \frac{28^{2/3}}{28^{2/3}-6} \approx 2.86$. Experiments show that the iteration now goes to $\approx 3.10$ in $N = 5$ steps, which is really nice.

• The equation $x - 3 = \frac{1}{x^2}$ can be smartly divided and multiplied to yield $x = \frac{1}{x(x-3)}$.
So $\varphi(x) = \frac{1}{x^2-3x}$, the iteration is $x_{k+1} = \frac{1}{x_k^2-3x_k}$.
Hopefully we will not hit a division by zero situaton.
Approximations are $x_0 = 1$ (given), $x_1 = -\frac{1}{2}$, $x_2 = \frac{4}{7}$.
$\varphi'(x) = \frac{3-2x}{x^2(x-3)^2}$, hence $|\varphi'(1)| = \frac{1}{4}$. This looks very hopeful, but the iteration then moves towards three and the formula for the derivative shows that $\varphi' \to \infty$ there, we have a similar problem on the other side around zero. This does not look very hopeful.
The experiments behave accordingly, showing a rather fast divergence.
Bonus: Relaxation with $\lambda = \frac{2}{3}$ uses the iterative function

$$\varphi_{2/3}(x) = \tfrac{2}{3} \cdot \tfrac{1}{x^2-3x} + \big(1 - \tfrac{2}{3}\big)x,$$

so $x_{k+1} = \frac{2}{3} \cdot \frac{1}{x_k^2-3x_k} + \big(1 - \frac{2}{3}\big)x_k$.
Therefore $x_1 = -\frac{1}{3} + \frac{1}{3} = 0$. This is not a good news, neither the equation nor $\varphi$ are defined there.
The optimal $\lambda$ for a neighborhood of $x_0 = 1$ satisfies $\varphi'_\lambda(1) = 0$. We have

$$\varphi'_\lambda(x) = \lambda\tfrac{3-2x}{x^2(x-3)^2} + (1-\lambda),$$

we substitute $x = 1$, set it equal to zero and obtain $\lambda = \frac{4}{3}$.
Experiments show that this does not save the iteration.
• From the equation $x - 3 = \frac{1}{x^2}$ we can also get to $x^2 - 3x = \frac{1}{x}$. Then $3x = x^2 - \frac{1}{x}$ and hence $x = \frac{1}{3}\big(x^2 - \frac{1}{x}\big)$.
So $\varphi(x) = \frac{1}{3}\big(x^2 - \frac{1}{x}\big)$, the iteration is $x_{k+1} = \frac{1}{3}\big(x_k^2 - \frac{1}{x_k}\big)$.
Approximations are $x_0 = 1$ (given), $x_1 = 0$. We encounter division by zero error when attempting to evaluate $x_2$, that's the end of the road.
Out of curiosity, $\varphi'(x) = \frac{1}{3}\big(2x + \frac{1}{x^2}\big)$, hence $|\varphi'(1)| = 1$, exactly on the border. We look at $|\varphi'(3)| > 2$, there is little point in trying to save this.
• From the equation $x - 3 = \frac{1}{x^2}$ we can also create $x^2 - 3x = \frac{1}{x}$. Then $x^2 = 3x + \frac{1}{x}$ and hence $x = \sqrt{3x + \frac{1}{x}}$.
So $\varphi(x) = \sqrt{3x + \frac{1}{x}}$, the iteration is $x_{k+1} = \sqrt{3x_k + \frac{1}{x_k}}$.
The root is risky, but the iterations were running over positive numbers so far, we will take the chance.
Approximations are $x_0 = 1$ (given), $x_1 = 2$.
$\varphi'(x) = \frac{3-\frac{1}{x^2}}{2\sqrt{3x+\frac{1}{x}}} = \frac{3x^2-1}{2x\sqrt{3x^3+1}}$, hence $|\varphi'(1)| = \frac{1}{2}$, this is a good news. To be on the safe side we also check on $|\varphi'(3)| = \frac{13}{6\sqrt{7}} \approx 0.48$, this is still good.
Experiments show convergence to the root $\approx 3.10$ after reasonable $N = 14$ steps.
Bonus: Relaxation with $\lambda = \frac{2}{3}$ uses the iterative function

$$\varphi_{2/3}(x) = \tfrac{2}{3} \cdot \sqrt{3x + \frac{1}{x}} + \big(1 - \tfrac{2}{3}\big)x,$$

so $x_{k+1} = \frac{2}{3} \cdot \sqrt{3x_k + \frac{1}{x_k}} + \big(1 - \frac{2}{3}\big)x_k$.
Therefore $x_1 = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$.
The optimal $\lambda$ for a neighborhood of $x_0 = 1$ satisfies $\varphi'_\lambda(1) = 0$. We have

$$\varphi'_\lambda(x) = \lambda\tfrac{3x^2-1}{2x\sqrt{3x^3+1}} + (1-\lambda),$$

we substitute $x = 1$, set it equal to zero and obtain $\lambda = 2$.
Experiments show that this value shortens the iteration to $N = 5$ steps, this is rather good.
If we try to optimize for a neighborhood of $x_0 = 3$, we get $\lambda \approx 1.92$, which is fairly close to our previous attempt with $\lambda = 2$. We try it and the run shortens to $N = 4$ steps, probably the best we can get here.

• From the equation $x - 3 = \frac{1}{x^2}$ we canalso get to $x^2 = \frac{1}{x-3}$, hence $x = \frac{1}{\sqrt{x-3}}$.
Considering that we want to start by substituting $x_0 = 1$ into the expression under the square root, we will not even look at this iteration.

That was all I could think of, but students surprised me again.

• Multiplying the equation $x - 3 = \frac{1}{x^2}$ we can also obtain $x^4 - 3x^3 = x$.
So $\varphi(x) = x^4 - 3x^3$, the iteration is $x_{k+1} = x_k^4 - 3x_k^3$.
Approximations are $x_0 = 1$ (given), $x_1 = -2$.
$\varphi'(x) = 4x^3 - 9x^2$, hence $|\varphi'(1)| = 5$, this looks quite bad. We also try $|\varphi'(3)| = 27$, this is again much larger than one. It is probablty not worth trying to save.
Bonus: Relaxation with $\lambda = \frac{2}{3}$ uses the iterative function

$$\varphi_{2/3}(x) = \tfrac{2}{3} \cdot (x^4 - 3x^3) + \bigl(1 - \tfrac{2}{3}\bigr)x,$$

so $x_{k+1} = \frac{2}{3}(x_k^4 - 3x_k^3) + \bigl(1 - \frac{2}{3}\bigr)x_k$.
Therefore $x_1 = -\frac{4}{3} + \frac{1}{3} = 1$.
The optimal $\lambda$ for a neighborhood of $x_0 = 1$ satisfies $\varphi'_\lambda(1) = 0$. We have

$$\varphi'_\lambda(x) = \lambda(4x^3 - 9x^2) + (1 - \lambda),$$

we substitute $x = 1$, set it equal to zero and obtain $\lambda = \frac{1}{6}$.
Experiments show that this value does lead to a convergent iteration, but extremely slow. Moreover, it converges to the number $x_f = 0$, which admittedly is a fixed point of the function $\varphi$, but it does not solve the original equation. We may get this problem when we multiply an equation by an ex in too high a power, then we sometimes add an artificial zero solution, which is exactly what happened here.
If we try to optimize for a neighborhood of $x_0 = 3$, we get $\lambda = -\frac{1}{26} \approx -0.04$. This one does save our iteration.