DEN: solutions for lab #10 (coronaversion)

1. a) From the first equation we isolat $y_2 = 2y_1 - y_1'$ $(*)$.

We substitute into the second and obtain

$$[2y_1 - y_1']' = 4y_1 - 3[2y_1 - y_1']$$
$$y_1'' + y_1' - 2y_1 = 0.$$

This is a classical homogeneous linear ODE. From the equation $\lambda^2+\lambda-2=0$ we obtain $\lambda = 1, -2$, hence the solution $y_1(x) = a\,e^x + b\,e^{-2x}$.

Substituting this into $(*)$ we then obtain $y_2(x) = a\,e^x + 4b\,e^{-2x}$.

b) We work with the matrix $A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$. Eigenvalues:

$$\det(A-\lambda E) = \begin{vmatrix} 2-\lambda & -1 \\ 4 & -3-\lambda \end{vmatrix} = (2-\lambda)(-3-\lambda)+4 = \lambda^2+\lambda-2 = 0.$$

Hence eigenvalues are $\lambda = 1, -2$. We find the associated eigenvectors:

$\underline{\lambda = 1}$: We solve the system

$$\begin{pmatrix} 2-1 & -1 \\ 4 & -3-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If we did not make any mistake, then the rows should be linearly dependent (and they are), so it is enough to solve one of the equations, for instance the one given by the first row: $v_1 - v_2 = 0$.

We can choose arbitrarily, but a zero vector is not allowed, so we take, say, $v_1 = 1$, then $v_2 = 1$. We have the eigenvector $\vec{v}_a = \binom{1}{1}$ and thus the solution

$$\vec{y}_a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{1\cdot x} = \begin{pmatrix} e^x \\ e^x \end{pmatrix}.$$

$\underline{\lambda = -2}$: In the system

$$\begin{pmatrix} 2+2 & -1 \\ 4 & -3+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we see dependence of rows again, which is as it should be. From the first row we have $4v_1 - v_2 = 0$. We need a solution $\neq \vec{0}$, so we choose, say, $v_1 = 1$, then $v_2 = 4$. We have an eigenvector $\vec{v}_b = \binom{1}{4}$ and hence the solution

$$\vec{y}_b = \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-2x} = \begin{pmatrix} e^{-2x} \\ 4e^{-2x} \end{pmatrix}.$$

A general solution: $\vec{y}(x) = a\vec{y}_a + b\vec{y}_b = \begin{pmatrix} a\,e^x + b\,e^{-2x} \\ a\,e^x + 4b\,e^{-2x} \end{pmatrix}$, $x \in I\!R$.

The problem was given in the traditional notation with individual equations, so it is proper to also state the answer in this way:

$$y_1(x) = a\,e^x + b\,e^{-2x},$$
$$y_2(x) = a\,e^x + 4b\,e^{-2x},\ x \in I\!R.$$

c) Initial values:

$$\left.\begin{aligned} y_1(0) &= 0 \\ y_2(0) &= -3 \end{aligned}\right\} \implies \left.\begin{aligned} a+b &= 0 \\ a+4b &= -3 \end{aligned}\right\} \implies \begin{aligned} b &= -1 \\ a &= 1 \end{aligned},$$

and we obtain $y_1(x) = e^x - e^{-2x}$, $y_2(x) = e^x - 4e^{-2x}$, $x \in I\!R$.

Remark: Elimination seems more convenient for most people, but it starts being troublesome already for systems 3×3, we will therefore use the matrix approach.

2. 1) General solution: The matrix is $A = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$.

Eigenvalues: $\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & -1 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda = 0$, hence $\lambda = 0, 3$.

$\underline{\lambda = 0}$: We are solving the system given by the matrix $\left(\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & 0 \end{array}\right)$, it is enough to solve $2v_1 - v_2 = 0$. The choice $v_1 = 1$ yields $v_2 = 2$. We have the solution $\vec{y}_a = \binom{1}{2} e^{0\cdot x} = \binom{1}{2}$.

$\underline{\lambda = 3}$: We are solving the system given by the matrix $\left(\begin{array}{cc|c} -1 & -1 & 0 \\ -2 & -2 & 0 \end{array}\right)$, it is enough to solve $-v_1 - v_2 = 0$.

The choice $v_1 = 1$ yields $v_2 = -1$. We have the solution $\vec{y}_b = \binom{1}{-1} e^{3x} = \binom{e^{3x}}{-e^{3x}}$.

A general solution: $\vec{y}(x) = \begin{pmatrix} a + b\,e^{3x} \\ 2a - b\,e^{3x} \end{pmatrix}$, $x \in I\!R$. We write it properly:

$$\begin{aligned} y_1(x) &= a + b\,e^{3x}, \\ y_2(x) &= 2a - b\,e^{3x}, \ x \in I\!R. \end{aligned}$$

2) Initial conditions:

$$\left.\begin{aligned} y_1(1) &= 1 + 2e^3 \\ y_2(1) &= 2 + e^3 \end{aligned}\right\} \implies \left.\begin{aligned} a + be^3 &= 1 + 2e^3 \\ 2a - be^3 &= 2 + e^3 \end{aligned}\right\} \implies \begin{aligned} a &= 1 + e^3 \\ b &= 1 \end{aligned},$$

hence $y_1(x) = 1 + e^3 + e^{3x}$, $y_2(x) = 2 + 2e^3 - e^{3x}$, $x \in I\!R$.

3. Matrix of the system: $A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$.

Eigenvalues: $\det(A - \lambda E) = \lambda^2 - 4\lambda + 5 = 0$, hence $\lambda = 2 \pm i$.

$\underline{\lambda = 2 + i:}$ We find the eigenvectors:

We should solve the system given by the matrix $\left(\begin{array}{cc|c} -1-i & 1 & 0 \\ -2 & 1-i & 0 \end{array}\right)$, the first row gives the equation $-(1+i)v_1 + v_2 = 0$. We choose $v_1 = 1$, then $v_2 = 1 + i$.

We have an eigenvector and use it to write one solution of the system:

$$\vec{u} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(2+i)t}.$$

We rewrite the exponential into the trig form $e^{2t+it} = e^{2t}[\cos(t) + i\sin(t)]$, multiply out with the vector and obtain

$$\vec{u}(t) = \begin{pmatrix} e^{2t}[\cos(t) + i\sin(t)] \\ e^{2t}[(\cos(t) - \sin(t)) + i(\cos(t) + \sin(t))] \end{pmatrix}.$$

Taking the real and imaginary parts we obtain two solutions for our basis:

$$\begin{aligned} \vec{u}_a(t) = \mathrm{Re}(\vec{u}) &= \begin{pmatrix} e^{2t}\cos(t) \\ e^{2t}[\cos(t) - \sin(t)] \end{pmatrix}, \\ \vec{u}_b(t) = \mathrm{Im}(\vec{u}) &= \begin{pmatrix} e^{2t}\sin(t) \\ e^{2t}[\cos(t) + \sin(t)] \end{pmatrix}. \end{aligned}$$

We have two vectors in our basis and the space of solutions is two-dimensional, so we do not need more. By the way, if we followed an analogous procedure with the eigenvalue $2 - i$, we would get the solutions $\vec{u}_a$ and $-\vec{u}_b$, that is, no new contributions to our basis.

A general solution is $\vec{x}(t) = a\vec{u}_a(t) + b\vec{u}_b(t)$, that is,

$$x_1(t) = a\,e^{2t}\cos(t) + b\,e^{2t}\sin(t),$$
$$x_2(t) = a\,e^{2t}[\cos(t) - \sin(t)] + b\,e^{2t}[\cos(t) + \sin(t)],\ \ t \in I\!R.$$

4. a) We rename $y_1 = y$. We obtain the equation $y_1''' + 3y_1'' - 2xy_1' - y_1 = 13$.

2) We reduce the order of derivative by one through the substitution $y_1' = y_2$. We get the equation $y_2'' + 3y_2' - 2xy_2 - y_1 = 13$.

3) We reduce the order of derivative by one through the substitution $y_2' = y_3$. We get the equation $y_3' + 3y_3 - 2xy_2 - y_1 = 13$.

We are happy with the order of derivative now.

4) We rewrite the equations: $\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_1 + 2xy_2 - 3y_3 + 13 \end{aligned}$

This system has the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2x & -3 \end{pmatrix}$.

b) We start by rewriting it as $y_1'''' + 13y_1''' - 23y_1'' + 14y_1' - 13y_1 = e^x$. Then we reduce:

$$\begin{aligned} y_1' &= y_2 \\ y_2''' + 13y_2'' - 23y_2' + 14y_2 - 13y_1 &= e^x \end{aligned} \implies \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3'' + 13y_3' - 23y_3 + 14y_2 - 13y_1 &= e^x \end{aligned}$$

$$\implies \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' + 13y_4 - 23y_3 + 14y_2 - 13y_1 &= e^x \end{aligned} \implies \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= 13y_1 - 14y_2 + 23y_3 - 13y_4 + e^x \end{aligned}$$

The matrix of the resulting system is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 13 & -14 & 23 & -13 \end{pmatrix}.$$