DEN: solutions for lab #11 (coronaversion)

1. System: $\begin{aligned} y_1' &= \qquad\quad y_2 \\ y_2' &= -9y_1 + 6y_2. \end{aligned}$

The matrix of the system is $A = \begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix}$. From the appropriate polynomial equation

$$0 = \det\begin{pmatrix} -\lambda & 1 \\ -9 & 6-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

we determine the double eigenvalue $\lambda = 3$.

We look up the right procedure and start searching for generalized eigenvectors.

The first eigenvector is found the usual way. The matrix of the system to solve is $\left(\begin{array}{cc|c} -3 & 1 & 0 \\ -9 & 3 & 0 \end{array}\right)$. From the equation $-3v_1 + v_2 = 0$ we find the eigenvector. I chose $v_1 = 1$ and obtained $v_2 = 3$, hence the eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Most people follow this choice.

The first contribution to our basis is the classical form $\vec{u}_a(x) = \vec{v}_1 e^{3x} = \begin{pmatrix} e^{3x} \\ 3e^{3x} \end{pmatrix}$.

The second generalized eigenvector is found using the system $\left(\begin{array}{cc|c} -3 & 1 & 1 \\ -9 & 3 & 3 \end{array}\right)$. In the first row equation $-3v_1 + v_2 = 1$ we can actually choose $v_1 = 0$, then $v_2 = 1$. We found $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Some people choose differently, good for them. Some popular choices include $\vec{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

The second solution for our basis is composed using a special formula:

$$\vec{u}_b(x) = [x\vec{v}_1 + \vec{v}_2]e^{3x} = \begin{pmatrix} x\,e^{3x} \\ (3x+1)e^{3x} \end{pmatrix}.$$

A general solution is $\vec{y}(x) = a\vec{u}_a + b\vec{u}_b$, that is,

$$\begin{aligned} y_1(x) &= a\,e^{3x} + b\,x\,e^{3x}, \\ y_2(x) &= 3a\,e^{3x} + b(3x+1)e^{3x}, \ x \in I\!R. \end{aligned}$$

We return to the original equation and see a solution $y(x) = a\,e^{3x} + b\,x\,e^{3x}$, it would definitely be easier to find it directly. Just a small check: Our transformation includes the formula $y_2 = y_1'$ and our functions satisfy this, which increases our confidence in the result.

2. a) Method of variation: We seek a solution of the form

$$\begin{aligned} y_{1p}(x) &= a(x) + b(x)e^{3x}, \\ y_{2p}(x) &= 2a(x) - b(x)e^{3x}. \end{aligned}$$

We substitute these into the given system and hope for some cancelling.

$$\begin{aligned} [a(x) + b(x)e^{3x}]' &= 2(a(x) + b(x)e^{3x}) - (2a(x) - b(x)e^{3x}) + 4e^{4x} \\ [2a(x) - b(x)e^{3x}]' &= -2(a(x) + b(x)e^{3x}) + (2a(x) - b(x)e^{3x}) + 2e^{x} \end{aligned}$$

$$\Longrightarrow \begin{aligned} a'(x) + b'(x)e^{3x} + 3b(x)e^{3x} &= 2a(x) + 2b(x)e^{3x} - 2a(x) + b(x)e^{3x} + 4e^{4x} \\ 2a'(x) - b'(x)e^{3x} - 3b(x)e^{3x} &= -2a(x) - 2b(x)e^{3x} + 2a(x) - b(x)e^{3x} + 2e^{x} \end{aligned}$$

$$\Longrightarrow \begin{aligned} a'(x) + b'(x)e^{3x} &= 4e^{4x} \\ 2a'(x) - b'(x)e^{3x} &= 2e^{x}. \end{aligned}$$

We readily solve this system, and integrate the resulting a' and b':

$$\begin{aligned} a'(x) &= \tfrac{1}{3}(2e^{x}+4e^{4x}) \implies a(x)=\tfrac{2}{3}e^{x}+\tfrac{1}{3}e^{4x},\\ b'(x)e^{3x} &= \tfrac{1}{3}(8e^{4x}-2e^{x}) \implies b'(x)=\tfrac{8}{3}e^{x}-\tfrac{2}{3}e^{-2x} \implies b(x)=\tfrac{8}{3}e^{x}+\tfrac{1}{3}e^{-2x}. \end{aligned}$$

Now we just substitute these into the variated functions:

$$\begin{aligned} y_{1p}(x) &= \left(\tfrac{2}{3}e^{x}+\tfrac{1}{3}e^{4x}\right)+\left(\tfrac{8}{3}e^{x}+\tfrac{1}{3}e^{-2x}\right)e^{3x}=e^{x}+3e^{4x},\\ y_{2p}(x) &= 2\left(\tfrac{2}{3}e^{x}+\tfrac{1}{3}e^{4x}\right)-\left(\tfrac{8}{3}e^{x}+\tfrac{1}{3}e^{-2x}\right)e^{3x}=e^{x}-2e^{4x}. \end{aligned}$$

We obtain a general solution

$$\begin{aligned} y_1(x) &= e^{x}+3e^{4x}+a+b\,e^{3x},\\ y_2(x) &= e^{x}-2e^{4x}+2a-b\,e^{3x},\ x\in I\!R. \end{aligned}$$

b) Method of undetermined coefficients: The right-hand sides include two types of expressions, with general forms $A\,e^{x}$ and $B\,e^{4x}$. None of them requires a correction, so we easily compose the following form of a solution:

$$\begin{aligned} y_{1p}(x) &= A\,e^{x}+B\,e^{4x},\\ y_{2p}(x) &= C\,e^{x}+D\,e^{4x}. \end{aligned}$$

We substitute these into the given system:

$$\begin{aligned} &[A\,e^{x}+B\,e^{4x}]' = 2(A\,e^{x}+B\,e^{4x})-(C\,e^{x}+D\,e^{4x})+4e^{4x}\\ &[C\,e^{x}+D\,e^{4x}]' = -2(A\,e^{x}+B\,e^{4x})+(C\,e^{x}+D\,e^{4x})+2e^{x}\\ \implies\ & \begin{aligned} A\,e^{x}+4B\,e^{4x} &= 2A\,e^{x}+2B\,e^{4x}-C\,e^{x}-D\,e^{4x}+4e^{4x}\\ C\,e^{x}+4D\,e^{4x} &= -2A\,e^{x}-2B\,e^{4x}+C\,e^{x}+D\,e^{4x}+2e^{x} \end{aligned}\\ \implies\ & \begin{aligned} (-A+C)e^{x}+(2B+D)e^{2x} &= 4e^{4x}\\ 2A\,e^{x}+(2B+3D)\,e^{2x} &= 2e^{x}. \end{aligned} \end{aligned}$$

Comparing opposite sides we obtain the equations $C-A=0$, $2B+D=4$, $2A=2$, $2B+3D=0$. We easily deduce that $A=C=1$, $D=-2$, $B=3$. We put these into our guess to obtain the particular solution $y_{1p}(x)=e^{x}+3e^{4x}$, $y_{2p}(x)=e^{x}-2e^{4x}$ that matches the one from variation, and hence we also obtain the same general solution.