DEN: lab #12 (coronaversion)

Today's topic is systems of linear algebraic equations, both elimination and iteration.

1. Consider the following system:

$$\left[\begin{aligned} 3x - 2y + z &= 7 \\ x - y \quad &= 2 \\ -4x + 2y \quad &= -6. \end{aligned}\right.$$

a) Write the extended matrix of the system and use the Gaussian elimination to reduce it to a row-echelon form. Then find the solution of the system using the backward substitution.
While doing the elimination, note down the steps that you perform. The form of those notes is up to you, for instance $(2) \leftrightarrow (3)$, $(2) - 3 \times (1) \mapsto (2)$, but it should be possible to recreate your elimination from them.

b) Determine the determinant of the matrix of the system, use the row-echelon form for it.

c) Explain what pivoting strategy you used. If you did not use partial pivoting, explain it and show how the elimination would go if partial pivoting was used (you do not have to actually do it, just indicate which row operations would be used). Explain what is the reason for using partial pivoting.

d) We are given a new system

$$\left[\begin{aligned} 3x - 2y + z &= 4 \\ x - y \quad &= 1 \\ -4x + 2y \quad &= -4 \end{aligned}\right.$$

that has the same left-hand sides as the previous one. Apply the row operations that you noted down in part a) just to the vector of the right-hand sides of the new systen. Connect the resulting vector with the reduced upper-triangular matrix from part a) and use the backward elimination to find the solution.
Check by substituting into the given system that you indeed found the right solution.

2. We are given the following system:

$$\left[\begin{aligned} x \quad + 2z &= -2 \\ 3x + y - z &= 9 \\ x + 4y + z &= 4. \end{aligned}\right.$$

a) Write the extended matrix of the system and use the Gaussian elimination to reduce it to a row-echelon form. Then find the solution of the system using the backward substitution.

b) Rewrite this system into a form suitable for the Jacobi iterative method. Given the initial vector $x_0 = y_0 = z_0 = 0$, calculate the next three iterations of the Jacobi iteration. Do you get the feeling that it would converge to the solution from part a)?

c) Apply the Gauss-Seidel iteration to this system. Namely, calculate the first two iterations given the initial vector $x_0 = y_0 = z_0 = 0$. Add a remark outlining the key difference compared to the Jacobi iteration at a suitable place.

3. We are given the system

$$\left[\begin{aligned} x \quad + 2z &= -2 \\ 3x + y - z &= 9 \\ x + 4y + z &= 4. \end{aligned}\right.$$

Incidentally, it is the one from the previous problem.

a) Reorder the system so that the Jacobi and Gauss-Seidel iterations have a high chance of success. Explain what shape of a system you are trying to achieve.

b) Apply the Jacobi iteration to the reordered system and calculate the next three iterations based on the initial vector $x_0 = y_0 = z_0 = 0$. Does it look hopeful?

c) Apply the Gauss-Seideliteration to the reordered system and calculate the next two iterations based on the initial vector $x_0 = y_0 = z_0 = 0$. Does it look hopeful?

• We know that the Gaussian elimination has asymptotic computatonal complexity of order 3. This would indicate that under ideal conditions, doubling the n would increase the runtime eight times.
However, things are not so simple in real life. The number of operations depends on the shape of the matrix (when there are zeros under the pivot, we save on operations), so comparison can be problematic when generating matrices randomly. We also did not include the algorithm overhead in our analysis, moreover, there can be differences in the way the algorithm is implemented on a particular machine, influencing key factors like access time to entries of our matrices. Finally, the runs are strongly influenced by the fact that popular operating systems like to do all kinds of things without asking.
That said, we could get at least some idea of time complexity by running a few experiments.

4. Assume that the run time of our program depends on the dimension n of the system according to the formula $T_n = cn^q$. We do not really care for the constant c, but we want to estimate q.
Derive a formula that would determine q based on two times T_n and T_{2n} obtained experimentally.

Bonus: In the Maple worksheet for this week you will find a code that generates a random matrix of the given dimension n and times the elimination for it. Then it does the same with matrix of dimension $2n$ and estimates q based on these two times using a formula that hopefully matches the one you derived above.
You can try it several times and write in your homework how it went. But do not overdo it with n, or your computer will be busy for a week.