DEN: solutions for lab #12 (coronaversion)

1. a) $\left(\begin{array}{ccc|c} 3 & -2 & 1 & 7 \\ 1 & -1 & 0 & 2 \\ -4 & 2 & 0 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 3 & -2 & 1 & 7 \\ -4 & 2 & 0 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right).$

Steps: $(1) \leftrightarrow (2)$, $(2) \mapsto (2) - 3 \times (1)$, $(3) \mapsto (3) + 4 \times (1)$, $(3) \mapsto (3) + 2 \times (2)$.

We get the system

$$\left[\begin{aligned} x - y \quad\;\; &= 2 \\ y + z &= 1 \\ 2z &= 4. \end{aligned}\right.$$

Backward substitution:

$$z = \tfrac{1}{2} \cdot 4 = 2; \qquad y = 1 - z = -1; \qquad x = 2 + y = 1.$$

Remark: Many students use "smart elimination" that I also like to use when eliminating by hand, but it can cause serious trouble with some applications (for instance the LUP decomposition). Popular offenders include tricks like multiplying the second row and adding the first one to it, or adding the second and the third row and put the outcome somewhere in the matrix while we work on the first column. The official GE allows for only two operations, namely switching rows and subtracting (adding) a multiple of a pivot row from a target row, these two operations are fine for all applications. However, note that the pivot row does not get multiplied itself during the latter operation, as multiplying rows is not allowed.

It is interesting to try elimination in the official way, then we can appreciate how hard computers have it, and how smart people can be.

b) The product of terms on the diagonal is 2. But we switched rows once, so the determinant is -2.

If you tried some tricks during elimination, you have to take them into account now.

c) In part a) we used pivoting suitable for calculations by hand. We prefer to have 1 as a pivot, in general we usually take the smallest possible number.

The partial pivoting chooses the largest candidate. If applied to the given system, we would start by swapping the first and the third row:

$$\left(\begin{array}{ccc|c} 3 & -2 & 1 & 7 \\ 1 & -1 & 0 & 2 \\ -4 & 2 & 0 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} -4 & 2 & 0 & -6 \\ 1 & -1 & 0 & 2 \\ 3 & -2 & 1 & 7 \end{array}\right).$$

Next we would add the first row multiplied by $\frac{1}{4}$ to the second one, and subtract the first row $\frac{1}{3}$ times from the third row.

d) We apply the operations from the first elimination:

$$\begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

We obtain the system

$$\left[\begin{aligned} x - y \quad\;\; &= 1 \\ y + z &= 1 \\ 2z &= 2. \end{aligned}\right.$$

The backward substitution:

$$z = 1; \qquad y = 1 - z = 0; \qquad x = 1 + y = 1.$$

This checks out.

Remark: This idea for saving work is the basis of the LUP method.

2. a) If we take it the easy way, we obtain

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 3 & 1 & -1 & 9 \\ 1 & 4 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & -7 & 15 \\ 0 & 4 & -1 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & -7 & 15 \\ 0 & 0 & 27 & -54 \end{array}\right).$$

We obtain a system that can be solved using the backward substitution.

$$\left[\begin{array}{l} x \quad + 2z = -2 \\ \quad y - z = 15 \\ \quad\; 27z = -54 \end{array}\right. \implies z = -2, \qquad y = 15 + z = 1, \qquad x = -2 - 2z = 2.$$

b) The standard way to iteration is

$$\left[\begin{array}{l} x = -2 - 2z \\ y = 9 - 3x + z \\ z = 4 - x - 4y. \end{array}\right.$$

We calculate:

$$\begin{array}{llllllll} x_0 = 0 & & x_1 = -2 - 2z_0 = -2 & & x_2 = -2 - 2z_1 = -10 & & x_3 = 58 \\ y_0 = 0 & \implies & y_1 = 9 - 3x_0 + z_0 = 9 & \implies & y_2 = 9 - 3x_1 + z_1 = 19 & \implies & y_3 = 9 \\ z_0 = 0 & & z_1 = 4 - x_0 - 4y_0 = 4 & & z_2 = 4 - x_1 - 4y_1 = -30 & & z_3 = -62. \end{array}$$

It does not seem likely that this would converge to the solution from part a).
c) The Gauss-Seidel iteration uses the same formulas as in part b):

$$\left[\begin{array}{l} x = -2 - 2z \\ y = 9 - 3x + z \\ z = 4 - x - 4y. \end{array}\right.$$

We calculate, and we always use the latest information:

$$\begin{array}{lll} x_0 = 0 & & x_1 = -2 - 2z_0 = -2 \\ y_0 = 0 & \implies & y_1 = 9 - 3x_1 + z_0 = 15 \\ z_0 = 0 & & z_1 = 4 - x_1 - 4y_1 = -54. \end{array}$$

The difference compared to the Jacobi iteration: we used the new values immediately after finding them.

$$\begin{array}{ll} & x_2 = -2 - 2z_1 = 106 \\ \implies & y_2 = 9 - 3x_2 + z_1 = -363 \\ & z_2 = 4 - x_2 - 4y_2 = 1350. \end{array}$$

This does not look any better.

3. a) When it comes to convergence of iterative methods, we appreciate systems with a strong diagonal. It is possible to rearrange the given system in this way.

$$\left[\begin{array}{l} 3x + y - z = 9 \\ x + 4y + z = 4 \\ x \quad\;\; + 2z = -2. \end{array}\right.$$

b) We get the formulas

$$\left[\begin{array}{l} x = 3 - \frac{1}{3}y + \frac{1}{3}z \\ y = 1 - \frac{1}{4}x - \frac{1}{4}z \\ z = -1 - \frac{1}{2}x. \end{array}\right.$$

We calculate:

$$\begin{array}{llll} x_0 = 0 & x_1 = 3 - \frac{1}{3}y_0 + \frac{1}{3}z_0 = 3 & x_2 = 3 - \frac{1}{3}y_1 + \frac{1}{3}z_1 = \frac{7}{3} & x_3 = 2 \\ y_0 = 0 \implies & y_1 = 1 - \frac{1}{4}x_0 - \frac{1}{4}z_0 = 1 \implies & y_2 = 1 - \frac{1}{4}x_1 - \frac{1}{4}z_1 = \frac{1}{2} \implies & y_3 = \frac{25}{24} \\ z_0 = 0 & z_1 = -1 - \frac{1}{2}x_0 = -1 & z_2 = -1 - \frac{1}{2}x_1 = -\frac{5}{2} & z_3 = -\frac{13}{6}. \end{array}$$

This could converge to $x = 2$, $y = 1$ and $z = -2$.

c) The Gauss-Seidel iteration uses the same formulas:

$$\begin{array}{lll} x_0 = 0 & x_1 = 3 - \frac{1}{3}y_0 + \frac{1}{3}z_0 = 3 & x_2 = \frac{25}{12} \\ y_0 = 0 \implies & y_1 = 1 - \frac{1}{4}x_1 - \frac{1}{4}z_0 = \frac{1}{4} \implies & y_2 = \frac{53}{48} \\ z_0 = 0 & z_1 = -1 - \frac{1}{2}x_1 = -\frac{5}{2} & z_2 = -\frac{49}{24}. \end{array}$$

These iterations also look like converging to $x = 2$, $y = 1$, $z = -2$.

4. We use the equations $T_n = cn^q$ and $T_{2n} = c2^q n^q$ to eliminate c (for instance we divide the second equation by the first) and obtain $2^q = \frac{T_{2n}}{T_n}$, hence

$$q = \log_2\left(\frac{T_{2n}}{T_n}\right) = \frac{\ln\left(\frac{T_{2n}}{T_n}\right)}{\ln(2)}.$$

Now for the experiment. I chased kids away from my computer and decided to use ten experiments for every given matrix size to make the outcome more reliable. In the chart you can see the range of times for the ten experiments and then the average time used in the formula for q.

n:	50	100	150	200
range(n) [sec]:	0.09—1.14	0.80—0.86	3.0—3.4	7.2—7.7
average(n) [sec]:	0.108	0.816	3.19	7.41
range($2n$) [sec]:	0.80—0.87	7.0—7.5	26.1—26.5	71.5—72.5
average($2n$) [sec]:	0.831	7.18	26.2	71.6
q:	2.95	3.14	3.03	3.27

The values of q are around three. The last experiment is a bit off, which may be related to the fact that matrices got significantly larger.

However, your mileage may vary. I used the opportunity of having three different versions of Maple (9.5, 14, 2018) on one computer, and conversely, I have version 14 on three computers that are 2, 7 and 13 years old. I tried the same experiment on all of these and each time I got somewhat different results, sometimes significantly. When I lecture this chapter, I always come to the lecture room earlier and test the computer and Maple there to see which n yields results reasonably close to three before I show this to students `;-)`.