DEN: lab #13 (coronaversion)

Today we will cover some remaining minor topics for the exam and do a review.

We know three methods for solving differential quations:

• separation for separable equations;

• the guessing method (of undetermined coefficients) for linear equations with constant coefficients, with right-hand side composed of special expressions;

• the variation method for linear equation with constant coefficients, but for first order equations we can also have non-constant coefficients as we can solve the homogeneous equation using separation then.

The restriction with the first and the third method is negative in a way: We cannot solve other types of equations with them, but we are also not sure that they will work for the right types, as it all depends on what integrals we get during the process. With the guessing method we know for sure that it will work for the right type of equation.

The final test will also include a problem of the following type:

1. Judge suitability of basic three solution methods for the following equations. Justify your answers.

a) $y' = 2y + 13e^{2x}$;

b) $y' = y^2 + 2x$;

c) $y' = \frac{y}{x} - \frac{1}{x}$;

d) $y'' = 4y + \arctan(x)$.

There will be also a few problems on the test where the solution is found based on understanding of notions and thinking rather than applying a memorized procedure. Here is one such problem.

2. Decide whether the following sets of functions could conceivably be fundamental systems for some homogeneous linear differential equation.

Justify your answers.

If your answer is yes, show one such equation.

a) $\{e^x, e^{2x}\}$;

b) $\{e^x, 1, x+1\}$;

c) $\{6, x+1, x-1\}$.

We conlcude this worksheet by reviewing the basic three methods.

3. Find a general solution of the equation $y' = \dfrac{y-1}{x}$.

4. Find a general solution of the equation $y' = 2y + 13e^{2x}$ using the guessing method.

5. Find a general solution of the equation $y' - \dfrac{1}{x}y = -\dfrac{1}{x}$ using the method of variation.

As a bonus we offer a selection of some problems that require thinking as a preparation for the exam. Do not send in your answers; you will find the solutions on the next page.

1. Create some differential equation (a true one, so that it features both y and its derivative) so that the function $y(x) = \sin(x) + \cos(x)$ is a solution of it.

2. For which values of the parameter p does the equation $y'' - 2py' + p^2 y = 0$ have no solution bounded on $(0, \infty)$ apart from the trivial one?

3. Is it possible to have a linear differential equation of order 1 with constant coefficients so that it has a solution whose asymptotic rate of growth at infinity is e^x, and also another one with asymptotic growth e^{3x}? If yes, how should it look like?

4. Consider the equation $y'' + y' - 2y = b(x)$. Find some right-hand side so that the solution of the resulting equation had the asymptotic growth $x^2 e^x$ at infinity.

Solution

1. SOlution using common sense:

Approach a) We find $y' = \cos(x) - \sin(x)$ and now we try to see y in it: $y' = \cos(x) + \sin(x) - 2\sin(x)$. So the equation $y' = y - 2\sin(x)$ will do.

Approach b) Choose some expression with y' and y, say, $y' + 13y$, and figure out the right-hand side:

$y' + 13y = \cos(x) - \sin(x) + 13\sin(x) + 13\cos(x)$.

The equation is $y' + 13y = 12\sin(x) + 14\cos(x)$.

A sophisticated solution: We create an equation with general solution $y(x) = a\sin(x) + b\cos(x)$, this requires $\lambda = \pm i$, that is, the characteristic equation should be $\lambda^2 + 1 = 0$. The equation $y'' + y = 0$ will do.

2. A general solution will be based on functions coming from characteristic numbers λ. The equation $\lambda^2 - 2p\lambda + p^2 = 0$ has solution

$$\tfrac{1}{2}\bigl(2p \pm \sqrt{4p^2 - 4p^2}\,\bigr) = p.$$

It is a real characteristic number of multiplicity two, so the general solution is $y(x) = a\,e^{px} + bx\,e^{px}$. For $a = b = 0$ we get a bounded trivial solution, this cannot be avoided, but all other solutions should be unbounded. We need $p > 0$, so this is the answer.

Remark: For $p = 0$ We would have the solution $y(x) = a + bx$, one could get $b = 0$ and suddenly we are getting non-trivial bounded solutions $y(x) = a$; we do not want that.

For $p < 0$ we combine the function e^{px} bounded on $(0, \infty)$, and another function. However, that one does not help much, since the choice $b = 0$ leads to the solution $y(x) = a\,e^{px}$, this is a non-trivial function bounded on $(0, \infty)$ and we do not want that. Just for completeness, the second function $x\,e^{px}$ is also bounded on $(0, \infty)$ for $p < 0$, we can deduce it from the fact that it is bounded on $\mathbb{R}$ and tends to zero at infinity.

3. A linear equation of order one with constant coefficients has the form $y' + \lambda y = b(x)$ and its solution is $y(x) = y_p + a\,e^{\lambda x}$, where y_p is a particular solution, and λ is a characteristic number. We can switch on and off the part $e^{\lambda x}$, so we need $\lambda = 3$. When $a \neq 0$, we satisfied one requirement. When $a = 0$, the part y_p becomes dominant. Therefore we say yes, it is possible, as long as the solution looks like $y(x) = e^x + a\,e^{3x}$. The equation should then be of the form $y' - 3y = A\,e^x$.

4. We have $\lambda = 1, -2$, the solution is therefore of the form $y(x) = a\,e^x + b\,e^{-2x} + y_p(x)$; the rate of growth x^2e^x must therefore be determined by a particular solution. This we determine by choosing the right-hand side based on our knowledge of the guessing method. Since $\lambda = 1$ is a characteristic number, we have a match here. Thus it is enough to take $b(x) = x\,e^x$ and the corrective term will increase the power to a square.