DEN: A trial exam

Find two hours in your busy schedule, set up a comfortable working environment, take a deap breath ... here we go.

1. a) Find the solution of the initial value problem

$$y'' - 5y' + 4y = 6e^x + 16x, \qquad y(0) = 6,\ y'(0) = 3.$$

b) Determine the symptotic rate of growth at infinity of a typical solution of the equation

$$y''' - 4y'' + 5y' - 2y = 0.$$

Hint: Characteristic numbers are $\lambda = 1, 1, 2$.

2. a) Find the solution of the initial value problem $\quad y' = 2\frac{\cos(x)}{\sin(x)}(y-1), \qquad y\left(\frac{\pi}{2}\right) = 2.$

b) (ver. 1) Find a general solution of the system $\quad \begin{aligned} y_1' &= 4y_1 - 2y_2 \\ y_2' &= 3y_1 - 3y_2 \end{aligned} \quad$ using the matrix approach.

b) (ver. 2) Use the method of variation to find a general solution of the equation $y' - 13y = 2e^{15x}$.

3. a) For the equation $\quad y' = e^y(1-y) \quad$ sketch its slope field, find stationary solutions and determine their stability.

b) For the equation $\quad y' = 13xy \quad$ investigate existence and uniqueness of solutions using the Picard theorem.

c) Judge suitability of basic three solution methods for the equation $\quad y' = xy + x^2$.

d) Deduce an approximating formula for the function e^{1+h} for small h with error $O(h^2)$.

4. a) Consider the integral $\int_1^7 \sqrt{|x-3|}\,dx$. Use the trapezoid method with partition size $n = 3$ to estimate (calculating by hand) its value. Explain the principle of this method with a picture (it need not be with the given function).

b) Derive the formula for the Newton root finding method, and explain it with a picture.

State the order of this method and explain what it means.

Grading guide

General remark: If you follow the correct procedure but a numerical error like $2+3=6$ happens (probably due to cosmic radiation) and it does not influence significantly the difficulty levels of the calculations that follow, then half a point is deducted for this error.

If a part of a problem is solved correctly, but with different input data (for instance due to an error in a previous part), then full points are awarded for this part.

1. a) [total: 15 points]

1. General solution.

A) Homogeneous version: $\lambda^2 - 5\lambda + 4 = 0 \implies \lambda = 1, 4$. $y_h(x) = a\,e^x + b\,e^{4x}$. [5 points]

B) Guessing method: $6e^x \implies A\,e^x$, $\lambda = 1$, correction, hence $Ax\,e^x$.

$16x \implies Bx + C$, $\lambda = 0$, no correction.

Guess: $y_p(x) = Ax\,e^x + Bx + C$ [4 points, -2 for error in correction].

Substitution into the equation leads to

$A(x+2)e^x - 5(A(x+1)e^x + B) + 4(Ax\,e^x + Bx + C) = -3A\,e^x + 4Bx + (-5B+4C) \overset{?}{=} 6e^x + 16x.$

From equations $-3A = 6$, $4B = 16$, $-5B + 4C = 0$ we get $A = -2$, $B = 4$, $C = 5$. [2 points]

General solution is $y(x) = 4x + 5 - 2x\,e^x + a\,e^x + b\,e^{4x}$, $x \in I\!R$. [1 point]

2. Initial values: We obtain $5 + a + b = 6$ and $2 + a + 4b = 3$. Thus $b = 0$, $a = 1$.

The desired solution is $y(x) = 4x + 5 - 2x\,e^x + e^x$, $x \in I\!R$.

[3 points, -1 when region of validity is missing]

b) [5 points]

General solution is $y(x) = a\,e^x + bx\,e^x + c\,e^{2x}$. At infinity we have $y(x) \sim c\,e^{2x}$; one could also say that a typical solution is of order $\Theta(e^{2x})$.

Remark: e^{2x} dominates $x\,e^x$, since $\dfrac{e^{2x}}{x\,e^x} = \dfrac{e^x}{x} \to \infty$.

2. a) [total: 10 points]

1. General solution: Restriction from the equation is $x \neq k\pi$.

Separation: We see the stationary solution $y(x) = 1$, $x \neq k\pi$. [-1 point if not mentioned]

For $y \neq 1$ we separate:

$$\int \frac{dy}{y-1} = 2\int \frac{\cos(x)}{\sin(x)}\,dx \implies \ln|y-1| = 2\ln|\sin(x)| + C \implies \ln|y-1| = \ln(|\sin(x)|^2) + C$$
$$\implies \ln|y-1| = \ln(\sin^2(x)) + C \implies |y-1| = e^{\ln(\sin^2(x))+C} = e^C e^{\ln(\sin^2(x))}$$
$$\implies |y-1| = e^C \sin^2(x) \implies y - 1 = \pm e^C \sin^2(x) \implies y = 1 + D\sin^2(x).$$

The choice $D = 0$ includes the stationary solution, so we have a general solution

$$y(x) = 1 + D\sin^2(x),\ x \neq k\pi.$$

[6 points, -1 if the region of validity is missing]

2. Initial condition yields $D = 1$, the choice of interval is determined by the initial time $x_0 = \frac{\pi}{2}$. Solution:

$$y(x) = 1 + \sin^2(x),\ x \in (0, \pi).$$

[4 points, -2 if the region of validity is missing]

b) (version 1) [total: 10 points]

From the matrix $\begin{pmatrix} 4-\lambda & -2 \\ 3 & -3-\lambda \end{pmatrix}$ we derive the equation

$$0 = (4-\lambda)(-3-\lambda) + 6 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

We find eigenvectors for the eigenvalues $\lambda = -2, 3$:

$\underline{\lambda = 3:}$ $\begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$ gives the equation $v_1 - 2v_2 = 0$ and eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We have the solution $\begin{pmatrix} 2e^{3x} \\ e^{3x} \end{pmatrix}$.

$\underline{\lambda = -2:}$ $\begin{pmatrix} 6 & -2 \\ 3 & -1 \end{pmatrix}$ gives the equation $3v_1 - v_2 = 0$ and eigenvector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. We have the solution $\begin{pmatrix} e^{-2x} \\ 3e^{-2x} \end{pmatrix}$.

[5 points]

General solution is $\vec{y}(x) = a \begin{pmatrix} 2e^{3x} \\ e^{3x} \end{pmatrix} + b \begin{pmatrix} e^{-2x} \\ 3e^{-2x} \end{pmatrix}$, that is,

$$y_1(x) = 2a\, e^{3x} + b\, e^{-2x},$$
$$y_2(x) = a\, e^{3x} + 3b\, e^{-2x}, \; x \in I\!R.$$

[5 points, −1 if the region of validity is missing]

b) (version 2) [total: 10 points]

A. Homogeneous equation: $y' - 13y = 0 \implies y_h(x) = C\, e^{13x}$. [4 points]

B. Variation: $y_p(x) = C(x)e^{13x}$. Substituting into the equation we get

$$C'(x)e^{13x} = 2e^{15x} \implies C'(x) = 2e^{2x} \implies C(x) = e^{2x}.$$

Thus $y_p(x) = e^{2x}e^{13x} = e^{15x}$ and a general solution is

$$y(x) = e^{15x} + C\, e^{13x}, \; x \in I\!R.$$

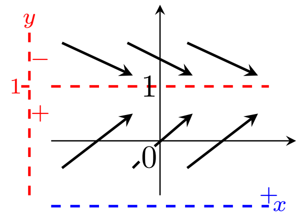
[6 points, −1 if the region of validity is missing]

3. a) [4 points]

Observations:

- The right-hand side does not feature x, so we can investigate stability.

- The right-hand side is a product, so we can investigate the influence of each factor separately.



The part e^y is always positive, no infuence, we ignore it.

The factor $1 - y$ has dividing curve $y = 1$ (horizontal line) and contributes − above it, + below. [3 points for a picture]

Stationary solutions? We are looking for y_0 that makes $e^{y_0}(1 - y_0) = 0$ regardless of x, we found $y_0 = 1$. This is an equilibrium, that is, $y(x) = 1$ is a stationary solution.

Stability: Based on the picture, $y_0 = 1$ is a stable equilibrium, that is, $y(x) = 1$ is a stable stationary solution. [1 point]

b) [4 points]

$\frac{\partial f}{\partial y} = 13x$, this goes to infinity when $x \to \pm\infty$, we have to prohibit this.

We therefore take x from $(-K, K)$ for $K > 0$, no restriction on y.

We claim that $\frac{\partial f}{\partial y} = 13x$ is bounded on rectangles $(-K, K) \times I\!R$, hence there are unique solutions passing through all points of these rectangles.

We pass $K \to \infty$ and reach our conclusion: There are unique solutions of the given equation passing through all points of $I\!R^2$.

c) [4 points]

Separation: $y' = x(x + y)$, can't separate.

Guessing method: No. The equation $y' - xy = x^2$ has its RHS of a suitable type, but its coefficients are not constants.

Variation: The equation is linear. Its coefficients are not all constant, but this is not a problem with first order equations. So variation is suitable for this equations, now everything depends on the integrals that will appear.

d) [8 points]

We use the Taylor expansion of the function $f(h) = e^{1+h}$ centered at $h = 0$.

We have $f^{(k)}(h) = e^{1+h}$, so $f^{(k)}(0) = e^{1+0} = e$. The Taylor expansion is

$$e^{1+h} = e + e(h-0) + \frac{1}{2!}e(h-0)^2 + \frac{1}{3!}e(h-0)^3 + \cdots$$

The approximation for small h with error $O(h^2)$ is

$$e^{1+h} = e + eh + O(h^2) \text{ or } e^{1+h} \approx e + eh.$$

4. a) [total: 10 points]

picture [2 points]

$h = \frac{7-1}{3} = 2$. Points $1, 3, 5, 7$.

$I \approx \frac{1}{2} \cdot 2 \cdot \left[\sqrt{2} + 2 \cdot \sqrt{0} + 2 \cdot \sqrt{2} + \sqrt{4}\right]$ [8 points].

Remark: This is how we like it, we can see what you were doing. Of course, we also do not mind answers like

$I \approx \frac{1}{2} \cdot 2 \cdot \left[\sqrt{2} + 2 \cdot \sqrt{0} + 2 \cdot \sqrt{2} + \sqrt{4}\right] = 3\sqrt{2} + 2.$

b) [total: 10 points]

Picture with a tangent line [1 point]

Formula $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$ [2 points]

Its derivation using the tangent line $y = f(x_k) + f'(x_k)(x - x_k)$ [4 points]

The Newton metoda is of order 2 [1 point]

This means $|E_{k+1}| \approx C|E_k|^2$, where E_k is the absolute error of the iteration x_k [2 points] (for E_k small and simple roots).