DEN: ODE – theoretical view: systems of equations

Definition.
By a **system of linear ODEs of order 1 with constant coefficients** we mean a system of the form

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_1(x) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_2(x) \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_n(x) \end{aligned}$$

where $b_i(x)$ are right hand-sides, $a_{ij} \in I\!R$ are coefficients.

An Initial Value Problem (IVP) or Cauchy problem for such a system has initial conditions
$y_1(x_0) = y_{1,0}$, $y_2(x_0) = y_{2,0}$, ... , $y_n(x_0) = y_{n,0}$.

The system is called **homogeneous** if $b_i(x) = 0$ for all $i = 1, \dots, n$.

Fact.
Every system of n linear ODEs of order 1 can be transformed via elimination to one linear ODE of order n.
Every system of n linear ODEs of order n_i can be transformed via elimination to one linear ODE of order $\sum n_i$.

Theorem. (on **existence and uniqueness** for systems)
Consider a system of linear ODEs of order 1.
If $b_i(x)$ are continuous on an open interval I, then for every $x_0 \in I$ and all $y_{1,0}, y_{2,0}, \dots, y_{n,0} \in I\!R$ there exists a solution of the corresponding IVP on I and it is unique.

Fact.
Every linear ODE of order n (and every system of linear ODEs with sum of orders n) can be equivalently transformed to a system of n linear ODEs of order 1.

A system

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_1(x) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_2(x) \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_n(x) \end{aligned}$$

can be written as $\vec{y}\,' = A\vec{y} + \vec{b}$, where $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ is the **matrix of the system**,
$\vec{b}(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_n(x) \end{pmatrix}$ is the **vector of RHS**, and $\vec{y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}$ is the unknown, then $\vec{y}\,' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$.

The system is homogeneous if $\vec{b} = \vec{0}$, where $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1}$.

Initital conditions are $\vec{y}(x_0) = \vec{y}_0$.

Theorem. (on **structure of solution set** for homogeneous systems)
Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$, where $A \in I\!R^{n\times n}$. The set of all solutions of this system on some open interval I is a linear space of dimension n.

Definition.
Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$, where $A \in I\!R^{n\times n}$.
By a **fundamental system of solutions** of this system on an open interval I we mean any basis of the space of its solutions on I.
If $\{\vec{y}_1, \dots, \vec{y}_n\}$ is a fundamental system of solutions, then we define its **fundamental matrix** on I by $Y(x) = \begin{pmatrix} \vec{y}_1(x) & \cdots & \vec{y}_n(x) \end{pmatrix}$ (an $n \times n$ matrix).

Fact.
Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$, where $A \in I\!R^{n\times n}$. If $Y(x)$ is its fundamental matrix on I, then a general solution of this system on I is $\vec{y}_h(x) = Y(x)\cdot\vec{c}$ for $\vec{c}\in I\!R^n$.

Theorem.
Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$, where $A \in I\!R^{n\times n}$.
Let $\vec{y}_1,\dots,\vec{y}_n$ be solutions of this system on an open interval I.
$\{\vec{y}_1,\dots,\vec{y}_n\}$ is a fundamental system of solutions of this system on I if and only if $\det(Y(x)) \neq 0$ on I, which is true if and only if $\det(Y(x_0)) \neq 0$ for some $x_0 \in I$.

Definition.
Let $A \in I\!R^{n\times n}$ be a matrix.
A number λ) is called an **eigenvalues** of A if there exists a non-zero vector $\vec{x}\in I\!R^n$ such that $A\vec{x} = \lambda\vec{x}$. Vectors $\vec{x}$ with this property are then called **eigenvectors** of A associated with (corresponding to) the eigenvalue λ.

Theorem.
Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$ with matrix $A \in I\!R^{n\times n}$.
If λ_0 is an eigenvalue of A with associated eigenvector $\vec{v}$, then $\vec{y} = \vec{v}\,e^{\lambda_0 x}$ is a solution of the given system on $I\!R$.
If $\lambda_1,\dots,\lambda_k$ are distinct eigenvalues of the matrix A, then the corresponding solutions form a linearly independent set.

Fact.
Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$ with matrix $A \in \mathbb{R}^{n\times n}$. Let λ_0 be an eigenvalue of A with associated eigenvector $\vec{v}$.
If λ_0 is a complex number, that is, $\mathrm{Im}(\lambda_0) \neq 0$, then $\mathrm{Re}(\vec{v}e^{\lambda_0 x})$ and $\mathrm{Im}(\vec{v}e^{\lambda_0 x})$ are linearly independent solutions of the given system on $\mathbb{R}$.

Fact.
Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$ with matrix $A \in \mathbb{R}^{n\times n}$. Let λ_0 be an eigenvalue of A of multiplicity m with associated eigenvector $\vec{v}$.
Consider vectors defined as follows:

$\vec{v}_1 = \vec{v}$,
$\vec{v}_2$ is a solution of $(A - \lambda_0 E_n)\vec{x} = \vec{v}_1$,
$\vec{v}_3$ is a solution of $(A - \lambda_0 E_n)\vec{x} = \vec{v}_2$,
$\vdots$
$\vec{v}_m$ is a solution of $(A - \lambda_0 E_n)\vec{x} = \vec{v}_{m-1}$.

Then the following functions are solutions of the given system on $\mathbb{R}$ and form a linearly independent set:

$$\begin{aligned}
\vec{y} &= \vec{v}_1 e^{\lambda_0 x},\\
\vec{y} &= \Big[\int (\vec{v}_1)\,dx + \vec{v}_2\Big] e^{\lambda_0 x} = (\vec{v}_1 x + \vec{v}_2)e^{\lambda_0 x},\\
\vec{y} &= \Big[\int (\vec{v}_1 x + \vec{v}_2)\,dx + \vec{v}_3\Big] e^{\lambda_0 x} = \big(\tfrac{1}{2}\vec{v}_1 x^2 + \vec{v}_2 x + \vec{v}_3\big)e^{\lambda_0 x},\\
&\vdots\\
\vec{y} &= \big(\tfrac{1}{(m-1)!}\vec{v}_1 x^{m-1} + \tfrac{1}{(m-2)!}\vec{v}_2 x^{m-2} + \cdots + \vec{v}_{m-1}x + \vec{v}_m\big)e^{\lambda_0 x}.
\end{aligned}$$

Theorem. (on **structure of solution set** for systems)
Consider a system of linear ODEs $\vec{y}\,' = A\vec{y} + \vec{b}(x)$. Let $\vec{y}_p$ be some solution of this system on I.
Then $\vec{y}_0$ is another solution of this system on I if and only if $\vec{y}_0 = \vec{y}_p + \vec{y}_h$ for some solution $\vec{y}_h$ of the system $\vec{y}\,' = A\vec{y}$ on I.
Thus, if $\vec{y}_h$ is a general solution of the associated homogeneous system on I, then $\vec{y}_p + \vec{y}_h$ is a general solution of the given system on I.

Algorithm (variation of parameters method)**.**
Given: a system $\vec{y}\,' = A\vec{y} + \vec{b}(x)$.
1. Find a general solution $\vec{y}_h$ of the associated homogeneous system $\vec{y}\,' = A\vec{y}$: $\vec{y}_h = Y(x)\cdot\vec{c}$, that is,
$y_{1h}(x) = c_1u_1(x) + c_2v_1(x) + \dots,$
$y_{2h} = \dots\,,$
$y_{nh}(x) = c_1u_n(x) + c_2v_n(x) + \dots$
2. a) Row variation: We seek solution of the form
$$y_1(x) = c_1(x)u_1(x) + c_2(x)v_1(x) + \dots,$$
$$\vdots$$
$$y_n(x) = c_1(x)u_n(x) + c_2(x)v_n(x) + \dots.$$
Unknown functions $c_i(x)$ are found by solving the system of equations
$$c_1'(x)u_1(x) + c_2'(x)v_1(x) + \dots = b_1(x),$$
$$\vdots$$
$$c_1'(x)u_n(x) + c_2'(x)v_n(x) + \dots = b_n(x).$$
From here determine (using e.g. elimination or Cramer rule) $c_1'(x),\dots,c_n'(x)$, integrating them one gets $c_1(x),\dots,c_n(x)$. Substitute these into modified $y_1,\dots,y_n$ to get $y_{1p},\dots,y_{np}$.
The general solution is $y_i = y_{ip} + y_{ih}$.

b) Vector variation: We seek solution of the form $\vec{y} = Y(x)\cdot\vec{c}(x)$.
Solve the equation $Y(x)\cdot\vec{c}\,'(x) = \vec{b}(x)$ for $\vec{c}\,'(x) = Y(x)^{-1}\vec{b}(x)$, integrating by rows get $\vec{c}(x)$ and substitute into $\vec{y}(x) = Y(x)\cdot\vec{c}(x)$. This yields $\vec{y}_p$, the general solution is then $\vec{y} = \vec{y}_p + \vec{y}_h$.