PARTIALLY ADDITIVE STATES ON ORTHOMODULAR POSETS

BY

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We fix a Boolean subalgebra $B$ of an orthomodular poset $P$ and study the mappings $s : P \to [0, 1]$ which respect the ordering and the orthocomplementation in $P$ and which are additive on $B$. We call such functions $B$-states on $P$. We first show that every $P$ possesses “enough” two-valued $B$-states. This improves the main result in [13], where $B$ is the centre of $P$. Moreover, it allows us to construct a closure-space representation of orthomodular lattices. We do this in the third section. This result may also be viewed as a generalization of [6]. Then we prove an extension theorem for $B$-states giving, as a by-product, a topological proof of a classical Boolean result.

1. Basic definitions and preliminaries.

1.1. Definition. An orthomodular poset (abbr. an OMP) is a triple $(P, \leq, ')$ such that

1. $(P, \leq)$ is a partially ordered set with a least element $0$ and a greatest element $1$,
2. the operation $' : P \to P$ is an orthocomplementation, i.e. for every $a, b \in P$ we have $a'' = a$ and $b' \leq a'$ whenever $a \leq b$,
3. the least upper bound exists for every pair of orthogonal elements in $P$ ($a, b \in P$ are orthogonal, $a \perp b$, if $a \leq b'$),
4. the orthomodular law is valid in $P$: $b = a \vee (b \wedge a')$ whenever $a \leq b$ ($a, b \in P$).

A typical example of an OMP is the lattice of all projections in a Hilbert space or, of course, a Boolean algebra. (We do not assume that $P$ is a lattice. If it is, we call it an orthomodular lattice.)

Throughout the paper, $P$ will be an arbitrary OMP and $B$ an arbitrary Boolean subalgebra of $P$. (By a Boolean subalgebra of $P$ we mean a subset of $P$ which forms a Boolean algebra with respect to $\leq$ and $'$ inherited from $P$, see also [4], [7].) Let us state our basic definition.

1.2. Definition. Let $B$ be a Boolean subalgebra of $P$. A partially additive state with respect to $B$ (abbr. a $B$-state) is a mapping $s : P \to [0, 1]$ such that
(1) if $a \leq b$ then $s(a) \leq s(b)$ ($a, b \in P$),
(2) $s(a') = 1 - s(a)$ ($a \in P$),
(3) $s(a \lor b) = s(a) + s(b)$ provided $a \perp b$ and $a, b \in B$.

Let us denote the set of all $B$-states on $P$ by $S_B(P)$. Thus $S_B(P) \subset [0, 1]^P$. In what follows we will make use of the following observations: The set $S_B(P)$ viewed as a subset of $[0, 1]^P$ is a convex compact. Indeed, the convexity is obvious and the compactness is a standard consequence of the Tikhonov theorem ($[0, 1]^P$ is considered with the pointwise topology). It should be noted that the “ordinary” state on $P$ is exactly an element of the intersection $\bigcap S_B(P)$, where $B$ runs over all Boolean subalgebras of $P$.

2. Two-valued $B$-states, $B$-ideals. Let us denote by $S_B^2(P)$ the set of all two-valued $B$-states on $P$. We will show in this section that $S_B^2(P)$ is rich enough to determine the ordering in $P$. This extends [13] which contains the same result in the much easier situation of $B$ being the centre of $P$.

Let us first introduce an auxiliary notion.

2.1. Definition. Let $B$ be a Boolean subalgebra of $P$. A partial ideal $I$ on $P$ with respect to $B$ (abbr. a $B$-ideal) is a nonempty subset of $P$ such that

(A) if $a \in I$ and $b \leq a$ then $b \in I$ ($a, b \in P$),
(B) $a \lor b \in I$ provided $a, b \in I \cap B$.

Further, we call a $B$-ideal $I$ proper if

(C1) $a \in I$ implies $a' \notin I$.

Finally, we call a proper $B$-ideal $I$ a $B$-prime ideal if

(C2) $a \in P \setminus I$ implies $a' \in I$.

In what follows we will sometime replace without noticing the condition (B) by the apparently weaker condition (B') equivalent to (B):

(B') $a \lor b \in I$ provided $a, b \in I \cap B$ and $a \perp b$.

The link between two-valued $B$-states and $B$-ideals is presented in the following simple proposition.

2.2. Proposition. There is a one-to-one correspondence between two-valued $B$-states and $B$-prime ideals given by the mapping $s \mapsto s^{-1}(0)$.

Proof. Obvious.

In the course of the following propositions we will show that any pair of noncomparable elements in $P$ is separated by a $B$-prime ideal.
2.3. Proposition. Let \( \{I_\alpha; \alpha \in A\} \) be a collection of \( B \)-ideals in \( P \). Then the least \( B \)-ideal containing all \( I_\alpha \) \( (\alpha \in A) \) is \( J = \bigcup \{I_\alpha; \alpha \in A\} \cup \{a \in P; a \leq b_1 \lor \cdots \lor b_n, \text{ where } b_k \in I_{\alpha_k} \cap B \text{ for any } k \in \{1, \ldots, n\}\} \).

Proof. The proof requires only a verification of the properties from the definition of a \( B \)-ideal.

Let us agree to call the \( B \)-ideal \( J \) from Proposition 2.3 the \( B \)-ideal generated by \( \{I_\alpha; \alpha \in A\} \).

Prior to the next propositions, observe that the elements \( b_k \) \( (k \in \{1, \ldots, n\}) \) in Proposition 2.3 can be chosen pairwise orthogonal.

2.4. Proposition. Let \( I \subset P \) be a proper \( B \)-ideal. Suppose that \( \{a, a'\} \cap I = \emptyset \) for an \( a \in P \). Then the \( B \)-ideal generated by \( \{I, [0, a]\} \) is proper.

Proof. Suppose that the \( B \)-ideal \( J \) generated by \( \{I, [0, a]\} \) is not proper and seek a contradiction. If \( J \) is not proper, then there is an \( e \in P \) such that \( \{e, e'\} \subset J \). Observe that \( \{e, e'\} \not\subset I \cup [0, a] \). Indeed, both \( e, e' \) cannot be in \( I \) and if \( e \in [0, a] \) then \( e' \not\in I \).

According to Proposition 2.3 we may assume that \( e \leq b_1 \lor b_2, b_1 \in I \cap B, b_2 \in [0, a] \cap B \) and \( b_1 \perp b_2 \). We may also assume without any loss of generality that \( e = b_1 \lor b_2 \). Hence \( e' \in J \cap B \) and therefore there are \( b_3 \in I \cap B, b_4 \in [0, a] \cap B \) such that \( b_3 \perp b_4 \) and \( e' = b_3 \lor b_4 \). Then \( b_1, b_2, b_3, b_4 \) are pairwise orthogonal and, moreover, \( 1 = e \lor e' = b_1 \lor b_2 \lor b_3 \lor b_4 \). Thus, \( a' \leq (b_2 \lor b_4)' = b_1 \lor b_3 \in I \), a contradiction.

2.5. Proposition. Each proper \( B \)-ideal is contained in a \( B \)-prime ideal.

Proof. By Zorn’s lemma, each proper \( B \)-ideal is contained in a maximal proper \( B \)-ideal. By Proposition 2.4, each maximal proper \( B \)-ideal is a \( B \)-prime ideal.

2.6. Proposition. Suppose that \( a \not\leq b \) \( (a, b \in P) \). Then there exists a \( B \)-prime ideal \( I \) such that \( a \not\in I \) and \( b \in I \).

Proof. By Proposition 2.4, the \( B \)-ideal generated by \( \{[0, b], [0, a']\} \) is proper. The rest follows from Proposition 2.5.

2.7. Theorem. Let \( B \) be a Boolean subalgebra of \( P \). Suppose that \( a \not\leq b \) \( (a, b \in P) \). Then there exists a two-valued \( B \)-state \( s \in S_B^2(P) \) such that \( s(a) = 1 \) and \( s(b) = 0 \).

Proof. This follows immediately from Propositions 2.6 and 2.2.

In the next section we will need the following result.
2.8. Theorem. Let \( B, B_1 \) be Boolean subalgebras of \( P \). Let \( s_1 \) be a two-valued state on \( B_1 \). Then there exists a two-valued \( B \)-state \( s \) on \( P \) such that \( s|B_1 = s_1 \).

Proof. Put \( I_1 = s_1^{-1}(0) \). Put further \( J = \{ b \in P; \text{there exists } a \in I_1 \text{ with } b \leq a \} \). Then \( J \) is a proper \( B \)-ideal and, according to Proposition 2.5, \( J \) is contained in a \( B \)-prime ideal \( I \). \( I_1 \) is a prime ideal on \( B_1 \), hence \( I \cap B = I_1 \). The rest follows from Proposition 2.2.

As the following example (due to Mirko Navara) shows, Theorem 2.7 cannot be improved in such a way that \( s \in S_{B_1}(P) \cap S_{B_2}(P) \) for given Boolean subalgebras \( B_1, B_2 \) of \( P \).

2.9. Example. Figure 1 shows the Greechie diagram (see [3]) of an orthomodular lattice \( P \). The elements \( a, b' \in P \) are not orthogonal, hence \( a \not\leq b \), but there is no \( s \in S_{B_1}(P) \cap S_{B_2}(P) \) such that \( s(a) = 1 \) and \( s(b) = 0 \).

3. A representation theorem for orthomodular lattices. The main result in this section is a representation of \( P \) by means of clopen sets in a compact Hausdorff closure space (a generalized Stone representation). We will show as an improvement of [6] (where \( B \) is the centre of \( P \)) that if \( P \) is a lattice and if we are given a Boolean subalgebra \( B \) in \( P \), we can ensure that the restriction of the representation to \( B \) becomes the Stone representation.

First we reformulate results of the previous section in a way convenient for our representation theorem.

3.1. Proposition. Let \( P \) be the set of all \( B \)-prime ideals in \( P \). Let the mapping \( i : P \to \exp P \) be defined by \( i(a) = \{ I \in P; a \notin I \} \). Finally, write \( \overline{A} = \bigcap\{ i(b); b \in P \text{ and } A \subset i(b) \} \) for any \( A \subset P \). Then

1) \( i(0) = \emptyset, i(1) = P \) and \( i : (P, \leq,') \to (i(P), \subset,') \) is an isomorphism,

2) if \( A_\alpha \in i(P) \) (\( \alpha \in A \)) and \( i : (P, \leq,') \to (i(P), \subset,') \) exists in \( (i(P), \subset,') \), then \( \bigvee_{\alpha \in A} A_\alpha = \bigcup_{\alpha \in A} A_\alpha \),

3) if \( A, B \in i(B) \) then \( A \vee B = A \cup B \) and \( A \wedge B = A \cap B \).
Proof. The first property follows from the definition of $i$ and from Theorem 2.7. As for the second property, we know that $\bigvee_{a \in A} A_a \in i(P)$ contains all $A_a$ ($a \in A$) and therefore $\bigcup_{a \in A} A_a \subseteq \bigvee_{a \in A} A_a$. Then the equality $\bigcup_{a \in A} A_a = \bigvee_{a \in A} A_a$ follows from the definition of the “bar” operation. Finally, suppose that $I \in i(a \lor b)$, $a, b \in B$. Then $a \lor b \notin I$ and therefore either $a \notin I$ or $b \notin I$. Hence $I \in i(a) \cup i(b)$. Thus, $i(a \lor b) = i(a) \cup i(b)$ and we have $A \lor B = A \lor B$. Dually, $A \land B = (A' \lor B')' = (A' \lor B')' = A \cap B$.

Prior to stating our main result in this section let us shortly review basic facts on closure spaces (see [2], [6]). By a closure space we mean a pair $(X, \overline{\cdot})$, where $X$ is a nonempty set and $\overline{\cdot}$ : $\exp X \to \exp X$ is an operation which has the following four properties:

1. $\overline{\emptyset} = \emptyset$,
2. $A \subseteq \overline{A}$ for any $A \subseteq X$,
3. $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$ ($A, B \subseteq X$),
4. $\overline{A \cup B} = \overline{A} \lor \overline{B}$ for any $A \subseteq X$.

A set $A \subseteq X$ is called closed in $(X, \overline{\cdot})$ if $\overline{A} = A$ and $B \subseteq X$ is called open if $X \setminus B$ is closed. A closure space $(X, \overline{\cdot})$ is called Hausdorff if any pair of points in $X$ can be separated by disjoint open sets, and $(X, \overline{\cdot})$ is called compact if any open covering of $X$ has a finite subcovering. It should be noted that the intersection of any collection of closed sets is again a closed set. However, the union of two closed sets need not be closed.

Let us agree to write $CO(X)$ for the collection of all subsets of $X$ which are simultaneously closed and open.

3.2. Theorem. Let $P$, $i$ and $\overline{\cdot}$ have the same meaning as in Proposition 3.1. Then $P$ is a compact Hausdorff closure space and $i(P) \subseteq CO(P)$. If $P$ is a lattice, then $i(P) = CO(P)$.

Proof. One verifies easily that $P$ is a closure space. Suppose that $a \in P$. Then $i(a) = \overline{i(a)}$ and therefore $i(a)$ is closed. Also, $i(a) = i(a') = i(a')'$ and therefore $i(a)$ is open. Thus $i(P) \subseteq CO(P)$. This allows us to prove that $(P, \overline{\cdot})$ is Hausdorff and compact. Indeed, if $I_1, I_2 \subseteq P$ and $I_1 \neq I_2$, then there is an element $a \in P$ such that $a \in I_1 \setminus I_2$ ($a' \in I_2 \setminus I_1$). We therefore have two disjoint open sets $i(a), i(a')$ which separate $I_1, I_2$.

To show that $P$ is compact, consider an open covering $\{A_\alpha; \alpha \in A\}$ of $P$. Since every closed set in $P$ is an intersection of elements of $i(P)$, every open set is a union of elements of $i(P)$, we therefore may (and will) suppose that $A_\alpha = i(a_\alpha)$ ($a_\alpha \in P$, $\alpha \in A$). Hence there is no $B$-prime ideal $I$ such that $I \supset \{a_\alpha; \alpha \in A\}$. This means that the $B$-ideal $J$ generated by $\{[0, a_\alpha]; \alpha \in A\}$ is not proper. It follows that for some $d \in P$ we have one of the following possibilities (see Proposition 2.3): Either $d \in [0, a_{\alpha_1}]$, $d' \in [0, a_{\alpha_2}]$ ($\alpha_1, \alpha_2 \in A$) or $d \leq b_1 \lor \cdots \lor b_n$ for $b_k \in B \cap [0, a_{\alpha_k}]$ ($\alpha_k \in A$, $5$
Thus, in both cases we have found a finite subcovering of \( A \). In the former case \( a'_{\alpha 1} \leq d' \leq a_{\alpha 2} \) and therefore \( \mathcal{P} = i(a_{\alpha 1}) \cup i(a'_{\alpha 1}) \subseteq i(a_{\alpha 1}) \cup i(a_{\alpha 2}) \). In the latter case we may (and will) assume the equality instead of the inequality. Thus, we have \( d \in B \). Hence \( d' \in J \cap B \) and therefore we can write \( d' = \tilde{b}_1 \lor \cdots \lor \tilde{b}_m \) (\( b_k \in [0, a_{\alpha k}] \cap B \), \( \alpha_k \in A \), \( k \in \{1, \ldots, m\} \)). Then we have

\[
\mathcal{P} = i(d \lor d') = i(b_1 \lor \cdots \lor b_n \lor \tilde{b}_1 \lor \cdots \lor \tilde{b}_m) \\
= i(b_1) \lor \cdots \lor i(b_n) \lor i(\tilde{b}_1) \lor \cdots \lor i(\tilde{b}_m) \\
\subseteq i(a_{\alpha 1}) \lor \cdots \lor i(a_{\alpha m}).
\]

Thus, in both cases we have found a finite subcovering of \( \{ A_\alpha; \alpha \in A \} \).

Suppose now that \( P \) is a lattice and \( A \in CO(\mathcal{P}) \). According to the definition of the closure operation we may write \( A = \bigcup_{\alpha \in A} i(a_\alpha) \) for some \( a_\alpha \in P \). Making use of the compactness of \( \mathcal{P} \) we have \( A = \bigcup_{k=1}^n i(a_{\alpha k}) \) (\( \alpha_k \in A \), \( k \in \{1, \ldots, n\} \)). Thus, \( A = \bigvee_{k=1}^n i(a_{\alpha k}) = i(\bigvee_{k=1}^n a_{\alpha k}) \in i(P) \).

Before we state our last result in this section, recall that a mapping \( f : L_1 \to L_2 \) between two orthomodular lattices is called orthoisomorphism if \( f \) is one-to-one and respects ordering and orthocomplementation.

3.3. Theorem. Let \( B \) be a Boolean subalgebra of an orthomodular lattice \( P \). Then there exists a compact Hausdorff closure space \( \mathcal{P} \) such that \( P \) is orthoisomorphic to \( CO(\mathcal{P}) \). Moreover, the orthoisomorphism \( f : P \to CO(\mathcal{P}) \) can be taken such that \( f(B) \) is the Stone representation of \( B \).

Proof. This follows from Theorems 3.2 and 2.8.

4. Extensions of \( B \)-states. It is obvious that a trace of a \( B \)-state on \( B \) is a state. It is natural to ask whether any state on \( B \) is a trace of a \( B \)-state, i.e. whether the restriction \( r : S_B(P) \to S(B) \) is onto. In Theorem 2.8 we have showed that this is true for two-valued states. Here we generalize this result to arbitrary states on \( B \).

4.1. Theorem. Let \( B, B_1 \) be Boolean subalgebras of \( P \). If \( s_1 \) is a state on \( B_1 \), then there exists a \( B \)-state \( s \) on \( P \) such that \( s|B_1 = s_1 \).

Proof. We use the compactness of \( S = S_B(P) \cap S_{B_1}(P) \). In some places we partially utilize the technique of [11] and [10].

Let \( s_1 \) be a state on \( B_1 \) and let \( D = \{ d_1, \ldots, d_n \} \) be a partition of \( B_1 \). Thus, \( \bigvee_{k=1}^n d_k = 1 \) and \( d_i \perp d_j \) for \( i \neq j \) (\( i, j \in \{1, \ldots, n\} \)). Put \( F_D = \{ s \in S; s|D = s_1|D \} \). Let \( D \) denote the set of all partitions of \( B_1 \). We will show that \( \mathcal{F} = \{ F_D; D \in \mathcal{D} \} \) is a filter base consisting of nonempty closed sets in \( S \). First, every set \( F_D \) is closed by the definition of the topology in \( S \) ("pointwise convergence"). Let now \( D_1, D_2 \) be two partitions of \( B_1 \). Then \( F_{D_1} \cap F_{D_2} \supset F_{D_1 \lor D_2} \), where \( D_1 \lor D_2 = \{ d_1 \lor d_2; d_1 \in D_1 \text{ and } d_2 \in D_2 \} \) is
a partition of $B_1$. Finally, let $D$ be a partition of $B_1$. For every $d \in D \setminus \{0\}$ take a state $s_d \in S_{B_1}^2(B_1)$ such that $s_d(d) = 1$ (Theorem 2.7). According to Theorem 2.8, for every $d \in D \setminus \{0\}$ there exists a $B$-state $\tilde{s}_d \in S_{B_1}^2(P)$ such that $\tilde{s}_d|B_1 = s_d$. Hence $\tilde{s}_d \in S$ and $s = \sum_{d \in D \setminus \{0\}} s_1(d) \tilde{s}_d \in F_D$. Thus, $F$ is a centred system. Since $S$ is compact, we have a $B$-state $s$ such that $s \in \bigcap F$. It follows immediately from the definition of $F$ that $s$ extends $s_1$. The proof is complete.

It may be of independent interest to note the following corollary of the previous result which might be viewed as a topological proof of a classical Boolean result (see [5], [11], compare also [8]).

4.2. Corollary. Let $B_1$ be a Boolean subalgebra of a Boolean algebra $B$. Then every state on $B_1$ extends over $B$.

5. Open questions. Another concept of partial additivity of states (also stronger than in [13]) is studied in [12] and [1], where a theorem analogous to Theorem 2.7 is proved. The definition of the so-called central state (abbr. $c$-state) differs from the definition of $B$-state in the third condition:

$$(3^c) \ s(a \lor b) = s(a) + s(b) \text{ provided } a \perp b \text{ and } a \in C(P), b \in P,$$

where $C(P)$ is the centre of $P$.

It is an open problem whether results analogous to those in this paper are valid for $B$-states that are simultaneously $c$-states.

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