1.1) Divide the polynomial \( p(x) = 2x^5 - x^4 + 4x^3 + 3x^2 - x + 1 \) by the polynomial \( q(x) = x^3 + x^2 - x + 1 \) and find the remainder.

1.2) Divide the polynomial \( p(x) = x^5 - 2x^4 - x^3 + x^2 - x + 2 \) by the polynomial \( q(x) = x^2 + 2x^2 + x + 1 \) and find the remainder.

1.3) Using Horner’s schema, find \( p(a) \) and \( p(b) \), where \( p(x) = 2x^4 - 3x^3 + 5x^2 - x + 5 \) and \( a = 3, b = \frac{1}{2} \).

1.4) Using Horner’s schema, find \( p(a) \) and \( p(b) \), where \( p(x) = 2x^4 - 3x^3 + 5x^2 - x + 5 \) and \( a = -2, b = 1 \).

1.5) Given \( p(x) = x^5 - 4x^3 - 2x^2 + 3x + 2 \), find all roots of \( p(x) \) together with their multiplicity and express the given polynomial as a product of real irreducible polynomials.

1.6) Given \( p(x) = x^5 + 8x^4 + 7x^3 - 52x^2 - 44x + 80 \), find all roots of \( p(x) \) together with their multiplicity and express the given polynomial as a product of real irreducible polynomials.

1.7) Find all solutions of \( 3x^3 - 7x^2 - 7x + 3 = 0 \).

1.8) Find all solutions of \( 2x^4 - 3x^3 - 4x^2 + 3x + 2 = 0 \).

1.9) Use Gaussian elimination to solve the following systems.

\[
\begin{align*}
1.9.1) & \quad x + 2y + z = 2 \\
& \quad -3x - 4y - 5z = -5 \\
& \quad 2x - y + 7z = 3,
\end{align*}
\]

\[
\begin{align*}
1.9.2) & \quad x + 2y + z = 2 \\
& \quad -3x - 5y - 5z = -5 \\
& \quad x + 4y + z = 0.
\end{align*}
\]

1.10) Use Gaussian elimination to solve the following systems.

\[
\begin{align*}
1.10.1) & \quad x - 3y - 2z = 6 \\
& \quad 2x - 4y - 3z = 8 \\
& \quad -3x + 6y + 8z = -5,
\end{align*}
\]

\[
\begin{align*}
1.10.2) & \quad x + 2y - 3z = 1 \\
& \quad 2x + 5y - 8z = 4 \\
& \quad 3x + 8y - 13z = 7.
\end{align*}
\]

1.11) Use Gaussian elimination to solve the following systems.

\[
\begin{align*}
1.11.1) & \quad x + y + 2z = 4 \\
& \quad 2x + 3y + 6z = 10 \\
& \quad 3x + 6y + 10z = 17,
\end{align*}
\]

\[
\begin{align*}
1.11.2) & \quad x_1 + x_2 - 2x_3 + 4x_4 = 5 \\
& \quad 2x_1 + 2x_2 - 3x_3 + x_4 = 3 \\
& \quad 3x_1 + 3x_2 - 4x_3 - 2x_4 = 1.
\end{align*}
\]

1.12) Use Gaussian elimination to solve the following systems.

\[
\begin{align*}
1.12.1) & \quad 2x + y - 3z = 2 \\
& \quad x - 3y - z = 3,
\end{align*}
\]

\[
\begin{align*}
1.12.2) & \quad x + 3y - 2z + 5t = 4 \\
& \quad 2x + 8y - z + 9t = 9 \\
& \quad 3x + 5y - 12z + 17t = 7.
\end{align*}
\]

1.13) Write down all solutions of the following system as \( p \in \mathbb{R} \) obtains all possible real values.

\[
\begin{align*}
2x - y - z &= p \\
-x - y &= p \\
x + 2y + z &= 1.
\end{align*}
\]

1.14) Write down all solutions of the following system as \( p \in \mathbb{R} \) obtains all possible real values.

\[
\begin{align*}
2x + y - 4z &= p^2 \\
-x + y + z &= -3 \\
x + 2y + z &= 1.
\end{align*}
\]
1.1) Divide the polynomial \( p(x) = 2x^5 - x^4 + 4x^3 + 3x^2 - x + 1 \) by the polynomial \( q(x) = x^3 + x^2 - x + 1 \) and find the remainder.

**Solution.** Polynomial long division is an algorithm for dividing a polynomial by another polynomial of the same or lower degree, it is a generalised version of the arithmetic technique called long division. Given any two polynomials \( p(x) \) (the dividend) and \( q(x) \) (the divisor) so that \( q(x) \) is not zero and the degree of \( q(x) \) is lower or equal to the degree of \( p(x) \), there exist and are uniquely determined a quotient \( s(x) \) and a remainder \( r(x) \) such that \( p(x) = s(x)q(x) + r(x) \), or \( \frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)} \). Moreover: either \( r(x) = 0 \) or the degree of \( r(x) \) is lower than the degree of \( q(x) \). For the given polynomials, we have:

\[
\begin{array}{ccc}
(2x^5 - x^4 + 4x^3 + 3x^2 - x + 1) & : & (x^3 + x^2 - x + 1) \\
-(2x^5 + 2x^4 - 2x^3 + 2x^2) & & 2x^2 - 3x + 9 \\
-3x^4 + 6x^3 + x^2 - x + 1 & & 9x^3 - 2x^2 + 2x + 1 \\
-(-3x^4 - 3x^3 + 3x^2 - 3x) & & -(9x^3 + 9x^2 - 9x + 9) \\
9x^3 - 2x^2 + 2x + 1 & & -11x^2 + 11x - 8 \\
\end{array}
\]

This means that \( r(x) = -11x^2 + 11x - 8 \) is the remainder, \( s(x) = 2x^2 - 3x + 9 \) is the quotient of the given polynomials, and

\[
\frac{2x^5 - x^4 + 4x^3 + 3x^2 - x + 1}{x^3 + x^2 - x + 1} = 2x^2 - 3x + 9 + \frac{-11x^2 + 11x - 8}{x^3 + x^2 - x + 1}.
\]

1.3) Using Horner’s schema, find \( p(a) \) and \( p(b) \), where \( p(x) = 2x^4 - 3x^3 - 5x^2 - 9x + 5 \) and \( a = 3, b = \frac{1}{2} \).

**Solution.** The Horner’s schema for \( p(3) \) reads:

\[
\begin{array}{cccccc}
2 & -3 & -5 & -9 & 5 \\
3 & & 6 & 9 & 12 & 9 \\
\end{array}
\]

It follows that \( p(3) = 14 \), and also that:

\[
2x^4 - 3x^3 - 5x^2 - 9x + 5 = (x - 3)(2x^3 + 3x^2 + 4x + 3) + 14,
\]

or equivalently:

\[
\frac{2x^4 - 3x^3 - 5x^2 - 9x + 5}{x - 3} = 2x^3 + 3x^2 + 4x + 3 + \frac{14}{x - 3}.
\]

The Horner’s schema for \( p\left(\frac{1}{2}\right) \) reads:

\[
\begin{array}{cccccc}
2 & -3 & -5 & -9 & 5 \\
\frac{1}{2} & & 1 & -1 & -3 & -6 \\
\end{array}
\]

It follows that \( p\left(\frac{1}{2}\right) = -1 \), and also that:

\[
2x^4 - 3x^3 - 5x^2 - 9x + 5 = (x - \frac{1}{2})(2x^3 - 2x^2 - 6x - 12) - 1,
\]

or equivalently:

\[
\frac{2x^4 - 3x^3 - 5x^2 - 9x + 5}{x - \frac{1}{2}} = 2x^3 - 2x^2 - 6x - 12 + \frac{-1}{x - \frac{1}{2}}.
\]
1.5) Given \( p(x) = x^5 - 4x^3 - 2x^2 + 3x + 2 \), find all roots of \( p(x) \) together with their multiplicity and express the given polynomial as a product of real irreducible polynomials.

Solution. All possible rational roots of the given polynomial (that has 1 as coefficient of the term with highest degree) belong to the set \( \{1, -1, 2, -2\} \), the possible integer divisors of the constant term, 2, of the polynomial. Thus we use Horner’s schema starting with \( a = 1 \):

\[
\begin{array}{cccccc}
1 & 0 & -4 & -2 & 3 & 2 \\
1 & 1 & -3 & -5 & -2 & 0 \\
\end{array}
\]

It follows that \( p(1) = 0 \), and

\[
(x^5 - 4x^3 - 2x^2 + 3x + 2) : (x - 1) = (x^4 + x^3 - 3x^2 - 5x - 2),
\]

or

\[
(x^5 - 4x^3 - 2x^2 + 3x + 2) = (x - 1)(x^4 + x^3 - 3x^2 - 5x - 2).
\]

Now we continue using the Horner’s schema with \( a = 1 \) for the polynomial \( q(x) = x^4 + x^3 - 3x^2 - 5x - 2 \), to see if 1 is a root of \( p(x) \) with multiplicity 2:

\[
\begin{array}{cccccc}
1 & 1 & -3 & -5 & -2 \\
1 & 2 & -1 & -6 & 2 \\
\end{array}
\]

This implies that \( q(1) = -8 \), thus 1 is a root of \( p(x) \) with multiplicity only one. We now use Horner’s schema with \( a = -1 \) for \( q(x) \), and we get:

\[
\begin{array}{cccccc}
1 & 1 & -3 & -5 & -2 \\
-1 & -1 & 0 & 3 & 2 \\
\end{array}
\]

This implies that \( q(-1) = p(-1) = 0 \), and

\[
(x^4 + x^3 - 3x^2 - 5x - 2) = (x + 1)(x^3 - 3x - 2),
\]

and

\[
(x^5 - 4x^3 - 2x^2 + 3x + 2) = (x - 1)(x + 1)(x^3 - 3x - 2).
\]

Now we continue using the Horner’s schema with \( a = -1 \) for the polynomial \( s(x) = x^3 - 3x - 2 \), to see if \(-1\) is a root of \( p(x) \) with multiplicity 2:

\[
\begin{array}{cccc}
-1 & 0 & -3 & -2 \\
1 & -1 & -2 & 0 \\
\end{array}
\]

Since \( s(-1) = 0 \), the root \(-1\) has multiplicity at least two for the given polynomial and:

\[
(x^5 - 4x^3 - 2x^2 + 3x + 2) = (x - 1)(x + 1)^2(x^2 - x - 2).
\]

Even if we can find the roots of the polynomial \( r(x) = x^2 - x - 2 \), using the quadratic formula, we prefer to continue checking the Horner’s schema again with \( a = -1 \) for the polynomial \( r(x) \), to verify if \(-1\) is a root of \( p(x) \) with multiplicity three:

\[
\begin{array}{cccc}
-1 & 0 & -3 & -2 \\
1 & -1 & -2 & 0 \\
\end{array}
\]

This implies that \( r(-1) = 0 \) and \(-1\) is a root of \( p(x) \) with multiplicity three:

\[
p(x) = (x^5 - 4x^3 - 2x^2 + 3x + 2) = (x - 1)(x + 1)^2(x^2 - x - 2).
\]

The given polynomial has simple roots 1 and 2, while \(-1\) is a root with multiplicity three.

1.7) Find all solutions of \( 3x^3 - 7x^2 - 7x + 3 = 0 \).

Solution. The given polynomial \( p(x) = 3x^3 - 7x^2 - 7x + 3 \) has constant term 3 and 3 is also the coefficient of the term with highest degree. Considering that the integer divisors of 3 are all the numbers \( \{1, -1, 3, -3\} \), all possible
rational roots of the polynomial \( p(x) \) belong to the set \( \{1, -1, 3, -3, \frac{1}{3}, -\frac{1}{3}\} \). We use Horner’s schema starting with \( a = 1 \):

\[
\begin{array}{ccc}
3 & -7 & -7 & 3 \\
1 & & & \\
3 & -4 & -11 & -8
\end{array}
\]

This implies that \( p(1) = -8 \), thus 1 is not a root of \( p(x) \). We now use Horner’s schema with \( a = -1 \) for \( p(x) \), and we get:

\[
\begin{array}{ccc}
-1 & 3 & -7 & -7 & 3 \\
3 & -10 & 3 & 0
\end{array}
\]

Thus \(-1\) is a root of \( p(x) \) and:

\[
3x^3 - 7x^2 - 7x + 3 = (x + 1)(3x^2 - 10x + 3).
\]

The remaining two roots can be found using the formula for quadratic equations, or again with a Horner’s schema, starting with the guess \( a = 3 \):

\[
\begin{array}{ccc}
3 & -10 & 3 \\
3 & -1 & 0
\end{array}
\]

Thus \(3\) is also a root of \( p(x) \) and:

\[
3x^3 - 7x^2 - 7x + 3 = (x + 1)(x - 3)(3x - 1).
\]

The solutions of the given equation are the roots of the polynomial \( p(x) \), thus \( x = -1 \), \( x = 3 \), and \( x = \frac{1}{3} \).

1.9) Use Gaussian elimination to solve the following systems.

1.9.1) \[
\begin{align*}
x + 2y + z &= 2 \\
-3x - 4y - 5z &= -5 \\
2x - y + 7z &= 3,
\end{align*}
\]

1.9.2) \[
\begin{align*}
x + 2y + z &= 2 \\
-3x - 5y - 5z &= -5 \\
x + 4y + z &= 0.
\end{align*}
\]

Solution. Gaussian elimination, or Gaussian reduction, is an algorithm for determining the solutions of a system of linear equations.

Given a linear system \( S, Ax = b \), of \( m \) equations with \( n \) unknowns \( x_1, x_2, \ldots, x_n \):

\[
\left\{ \begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m.
\end{array} \right.
\]

we indicate with \( R_i \) the \( i \)-th equation of the system: \( R_i : a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i \)

We define the following elementary row operations:

\( (E_1) \). (Row interchange) Interchange two rows, i.e. interchange \( R_i \) and \( R_j \), \( R_i \leftrightarrow R_j \).

\( (E_2) \). (Row scaling) Multiply a non-zero constant throughout a row, i.e. multiply \( R_k \) by \( c \neq 0 \) thus replace \( R_k \) by \( cR_k \).

\( (E_3) \). (Row addition) Replace a row by itself plus a constant multiple of another row, replace \( R_k \) by \( R_k + cR_j \).

Theorem If a linear system \( S_1 \) is obtained from a linear system \( S \) using elementary row operation, then \( S_1 \) and \( S \) are equivalent, i.e. they have the same set of solutions.

Gaussian elimination is a method that consists in finding appropriate row operations to reduce the given system to subsequently easier ones whose solution can be directly found.

1.9.1) We will use the symbol \( \sim \) to indicate the equivalence of systems.

\[
\begin{align*}
x + 2y + z &= 2 & \sim & R_2 + 3R_1 & x + 2y + z &= 2 \\
-3x - 4y - 5z &= -5 & & 2y - 2z &= 1 \\
2x - y + 7z &= 3 & \sim & R_3 - 2R_1 & -5y + 5z &= -1 \\
& & & \sim & 2y - 2z &= 1 \\
& & & & 0 &= 3.
\end{align*}
\]
Solution. 1.11.1) We will use the symbol \( R \) to indicate the equivalence of systems.

\[
\begin{align*}
    x + 2y + z &= 2 \\
    -3x - 5y - 5z &= -5 \\
    x + 4y + z &= 0
\end{align*}
\]

Using Gaussian elimination, we can obtain:

\[
\begin{align*}
    R_2 + 3R_1 & \Rightarrow x + 2y + z = 2 \\
    y - 2z &= 1 \\
    R_3 - R_1 & \Rightarrow 2y = -2.
\end{align*}
\]

At this point, we can already get the solution of the system, starting from the last equation (\( y = -1 \)) and substituting into the second and subsequently the first equation (we get \( z = -1, \ x = 5 \)).

1.11.2) We now solve the following:

\[
\begin{align*}
    x + 2y + z &= 2 \\
    y - 2z &= 1 \\
    z &= -1
\end{align*}
\]

At this point we see that the system has a solution, and we continue to perform row operations till the result can be read directly from the final system:

\[
\begin{align*}
    R_3 - 2R_2 & \Rightarrow x + 2y + z = 2 \\
    4z &= -4 \\
    y - 2z &= 1 \\
    R_2 + 2R_3 & \Rightarrow y = -1 \\
    R_1 - R_3 & \Rightarrow z = -1.
\end{align*}
\]

1.11) Use Gaussian elimination to solve the following systems.

1.11.1) \( x + y + 2z = 4 \)

1.11.2) \( x_1 + x_2 - 2x_3 + 4x_4 = 5 \)

Solution. 1.11.1) We will use the symbol \( \sim \) to indicate the equivalence of systems.

\[
\begin{align*}
    x + y + 2z &= 4 \\
    2x + 3y + 6z &= 10 \\
    3x + 6y + 10z &= 17
\end{align*}
\]

At this point we see that the system has a solution, and we continue to perform row operations till the result can be read directly from the final system:

\[
\begin{align*}
    R_2 - R_3 & \Rightarrow x = 2 \\
    R_1 - R_2 & \Rightarrow y = 1 \\
    R_3/2 & \Rightarrow z = \frac{1}{2}.
\end{align*}
\]

1.11.2) We now solve the system:

\[
\begin{align*}
    x_1 + x_2 - 2x_3 + 4x_4 &= 5 \\
    2x_1 + 2x_2 - 3x_3 + x_4 &= 3 \\
    3x_1 + 3x_2 - 4x_3 - 2x_4 &= 1
\end{align*}
\]

In this case we say that the pivot variables are \( x_1 \) and \( x_3 \), while \( x_2 \) and \( x_4 \) are free variables. We may continue the Gaussian reduction with row operation \( R_1 + 2R_2 \), and get:

\[
\begin{align*}
    x_1 + x_2 - 10x_4 &= -9 \\
    x_3 - 7x_4 &= -7
\end{align*}
\]

Now the infinitely many solutions of the given system can be expressed as:

\[
\begin{align*}
    x_1 &= -\alpha - 10\beta - 9 \\
    x_2 &= \alpha \\
    x_3 &= -7\beta - 7 \\
    x_4 &= \beta,
\end{align*}
\]

where \( \alpha, \beta \in \mathbb{R} \) are any possible real number.

1.13) Write down all solutions of the following system as \( p \in \mathbb{R} \) obtains all possible real values.

\[
\begin{align*}
    2x - y - z &= p \\
    -x - y &= p \\
    -x + 2y + z &= 1.
\end{align*}
\]
Solution. We start a Gaussian elimination with row operations \(-R_2 \leftrightarrow R_1, R_1 + 2R_2\) and \(R_3 - R_2\), so to get:

\[
\begin{align*}
  x + y &= -p \\
-3y - z &= 3p \\
3y + z &= 1 - p
\end{align*}
\]

\[\sim \]

\[
\begin{align*}
  x + y &= -p \\
 3y + z &= -3p \\
0 &= 1 + 2p
\end{align*}
\]

We now start a discussion: if \(1 + 2p \neq 0\), the system has no solutions because the last equation is inconsistent; while if \(1 + 2p = 0\), i.e. \(p = -\frac{1}{2}\), then the last equation is \(0 = 0\), and can be disregarded (as it is satisfied for all possible values of \(x, y\) and \(z\)). Thus the given system is equivalent to:

\[
\begin{align*}
  x + y &= \frac{1}{2} \\
3y + z &= \frac{3}{2}
\end{align*}
\]

The infinitely many solutions of the system can be expressed with the use of a parameter: choosing \(y\) as a free variable, \(y = \alpha, \alpha \in \mathbb{R}\), we have that for every possible \(\alpha \in \mathbb{R}\),

\[
\begin{align*}
  x &= \frac{1}{2} - \alpha \\
  y &= \alpha \\
  z &= \frac{3}{2} - 3\alpha
\end{align*}
\]

is a solution of the given system where \(p = -\frac{1}{2}\).
2.1) Given \( \mathbf{A} = \begin{pmatrix} 3 & 1 & -2 \\ 4 & 0 & 2 \end{pmatrix} \), \( \mathbf{B} = \begin{pmatrix} -2 & 0 & 1 \\ 6 & 4 & -1 \end{pmatrix} \), calculate \( \mathbf{C} = 2\mathbf{A} - 3\mathbf{B} \).

2.2) Given \( \mathbf{v}_1 = (-1, 1, 1), \mathbf{v}_2 = (1, 0, -3), \mathbf{v}_3 = (2, 4, 5) \), calculate \( \mathbf{x} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + 4\mathbf{v}_3 \).

2.3) Determine if the vector \( \mathbf{v} = (2, -5, 3) \) in \( \mathbb{R}^3 \) is a linear combination of \( \mathbf{u}_1 = (1, -3, 2), \mathbf{u}_2 = (2, -4, -1) \) and \( \mathbf{u}_3 = (1, -5, 7) \).

2.4) Write \( \mathbf{x} = (3, -1, 5) \) as a linear combination of \( \mathbf{v}_1 = (-1, 1, 1), \mathbf{v}_2 = (1, -1, 1), \mathbf{v}_3 = (1, 1, -1) \).

2.5) Determine if the matrix \( \mathbf{M} = \begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix} \) is a linear combination of \( \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix} \).

2.6) In \( \mathcal{M}^{2-2} \) show that \( \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \) are Linearly Independent.

2.7) In \( \mathcal{M}^{2-2} \) show that \( \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} \) and \( \mathbf{C} = \begin{pmatrix} 1 & -4 \\ 2 & -5 \end{pmatrix} \) are Linearly Dependent.

2.8) Prove that \( \mathbf{v}_1 = (5, 4, 3), \mathbf{v}_2 = (3, 3, 2), \mathbf{v}_3 = (8, 1, 3) \) are linearly dependent.

2.9) Prove that \( \mathbf{v}_1 = (2, -3, 1), \mathbf{v}_2 = (3, -1, 5), \mathbf{v}_3 = (1, -4, 3) \) are linearly independent.

2.10) Determine for what values of \( \alpha \in \mathbb{R} \) the vectors \( \mathbf{v}_1 = (1, \alpha, 1), \mathbf{v}_2 = (0, 1, \alpha), \mathbf{v}_3 = (\alpha, 1, 0) \) are linearly independent.

2.11) For what values of \( \lambda \in \mathbb{R} \) are the polynomials \( p_1(t) = 2 + 5t + t^2, p_2(t) = 4 + 10t + 8t^2, p_3(t) = 6 + 10t + \lambda^2 \) linearly dependent?

2.12) Given \( \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 & 2 \\ 3 & 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 & 2 \\ -3 & d \end{pmatrix} \), find \( d \) such that \( \mathbf{A}, \mathbf{B}, \mathbf{C} \) are linearly dependent.

2.13) Given \( \mathbf{A} = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ -1 & -8 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 & 4 \\ -4 & 2 \end{pmatrix} \), find \( d, e, f \) so that \( \mathbf{A}, \mathbf{B}, \mathbf{C} \) are linearly dependent.

2.14) In \( \mathcal{P}_2(t) \) prove that \( p_1(t) = 1 + t^2, p_2(t) = 2 + t - t^2, p_3(t) = 1 - t \) are linearly independent.

2.15) Given \( g = 5t - 7t^2 \) in \( \mathcal{P}(t) \), show that \( g \in \text{span}\{g_1, g_2, g_3\} \) where \( g_1 = 1 + t - 2t^2, g_2 = 7 - 8t + 7t^2, g_3 = 3 - 2t + t^2 \).

2.16) In the linear space of polynomial functions \( \mathcal{P}(t) \) are given \( p_1(t) = 2 + t, p_2(t) = 2 + t^2, p_3(t) = 2 + t^3, q(t) = 1 - 2t + t^2 - t^3 \). Prove that \( q \notin \text{span}\{p_1, p_2, p_3\} \).

2.17) Prove that in any linear space if \( \mathbf{x} \) and \( \mathbf{y} \) are linearly independent then \( \mathbf{x} + \mathbf{y} \) and \( \mathbf{x} - \mathbf{y} \) are two linearly independent vectors.

2.18) Prove that in any linear space if \( \mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0} \) then \( \text{span}\{\mathbf{x}, \mathbf{y}\} = \text{span}\{\mathbf{y}, \mathbf{z}\} \).

2.19) Given \( \mathbf{u}_1 = (1, 1, 1), \mathbf{u}_2 = (1, 2, 3) \) and \( \mathbf{u}_3 = (1, 5, 8) \), show that \( \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \mathbb{R}^3 \).

2.20) Prove that the set \( B = \{(1, 0, 1), (1, 0, 0), (1, 1, 1)\} \) is a basis of \( \mathbb{R}^3 \). Find the coordinates of \( (2, 3, -1) \) with respect to the ordered basis \( B \).

2.21) Given \( p_1(t) = 1, p_2(t) = t - 3, p_3(t) = (t - 1)(t + 2), \) show that \( B = \{p_1, p_2, p_3\} \) is a basis of the space \( \mathcal{P}_2(t) \) of all polynomial functions with degree \( \leq 2 \). Find the coordinates of \( p(t) = 2t^2 + t + 2 \) with respect to \( B \).

2.22) Show that \( \mathbf{v}_1 = (2, 1, -3), \mathbf{v}_2 = (3, 2, -5), \mathbf{v}_3 = (1, -1, 1) \) form a basis of \( \mathbb{R}^3 \). Find coordinates of \( \mathbf{v} = (7, 6, -14) \) with respect to this basis.

2.23) Given \( p_1 = 1 - x^2, p_2 = 1 + x, p_3 = -1 + x + x^3 \), extend \( \{p_1, p_2, p_3\} \) to a basis of \( \mathcal{P}_3 \). (Use coordinates)
2.24) Show that if \( \{v_1, v_2, v_3, v_4\} \) is a basis of a linear space \( L \) then \( \{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\} \) is also a basis of \( L \). (Use coordinates)

2.25) Given the linear space \( \mathcal{P}(t) \) of all polynomials, determine if the following are linear subspaces
   i) all polynomials with integer coefficients
   ii) all polynomials with even powers of \( t \) (i.e. containing only terms of the type \( t^{2n} \) where \( n=0,1,2,... \))
   iii) all polynomials with degree greater or equal to six.

2.26) Is \( W = \{(a,b,c) : a \geq 0\} \) a linear subspace of \( \mathbb{R}^3 \)?

2.27) In the linear space \( F \) of all real functions \( F = \{f : \mathbb{R} \to \mathbb{R}\} \), decide if the sets defined below form linear subspaces or not.
   \( M_1 = \{f \in F : f(1) = 0\} \)
   \( M_2 = \{f \in F : f(3) = 1\} \)

2.28) Consider the subsets of \( \mathbb{R}^n \) defined below and for each of them prove if it is a linear subspace of \( \mathbb{R}^n \) or not:
   \( M_1 = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 = x_n\} \)
   \( M_2 = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i \in \mathbb{Z}, i = 1, ..., n\} \)
   \( M_3 = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 0\} \)
   \( M_4 = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 = 2\} \)
   \( M_5 = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_k = 0 \text{ for } k \text{ even}\} \)
   \( M_6 = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 \leq 0\} \)

2.29) Given the following subsets of \( \mathcal{P}_3(t) \) determine if they are linear subspaces of \( \mathcal{P}_3(t) \) or not. In affirmative case, find the dimension and a basis for each subspace:
   \( L_1 = \{p \in \mathcal{P}_3 : p(0) = 0\} \)
   \( L_2 = \{p \in \mathcal{P}_3 : p(-t) = p(t)\} \)

2.30) Given the following subsets of \( \mathbb{R}^3 \) determine if they are linear subspaces of \( \mathbb{R}^3 \) or not. In affirmative case, find the dimension and a basis for the subspace:
   \( M_1 = \{(x,y,z) : 2x - y + z = 2\} \)
   \( M_2 = \{(x,y,z) : x - 2y - z = 0\} \)

2.31) a) Prove that if \( M \) and \( N \) are subspaces of a vector space \( L \) then \( M \cap N \) is a subspace of \( L \).
   b) Given \( M = \text{span}\{(1,2,1,0),(-1,1,1,1)\} \) and \( N = \text{span}\{(1,2,1,-2),(2,1,0,1)\} \) (subspaces of \( \mathbb{R}^4 \)), find the dimension and a basis for \( M \cap N \).
   c) Consider the space \( U = \text{span}\{M \cup N\} \), find the dimension and a basis of \( U \).

2.32) In \( \mathbb{R}^4 \) are given the vectors \( v_1 = (0,3,-2,4), v_2 = (0,1,-1,3), v_3 = (0,0,1,-5), v_4 = (0,5,-4,10) \). Find a basis of the space \( M = \text{span}\{v_1, v_2, v_3, v_4\} \). Show that \( B = \{(0,2,-1,1),(0,1,0,-2)\} \) is also a basis of \( M \).

2.33) Consider \( M = \{p(t) \in \mathcal{P}^3(t) : p(t) \text{ is divisible by } (t-1)\} \). Prove that \( M \) a linear subspace of \( \mathcal{P}^3 \), find a basis and dimension of \( M \).

2.34) Given the equation \( x + 2y - z + 5w = 0 \) prove that the set \( S \) of all its solutions is a subspace of \( \mathbb{R}^4 \). Find the dimension and a basis of \( S \).

2.35) Consider \( Q = \{p(x) \in \mathcal{P}^4(x) : p(x) = x^4 + p\left(\frac{1}{2}\right) \} \) for any \( x \in \mathbb{R}, x \neq 0 \). Prove that \( Q \) a linear subspace of \( \mathcal{P}^4 \), find a basis and dimension of \( Q \).

2.36) Consider the subset of \( \mathcal{M}^{2,2} \) formed by the symmetric matrices \( a_{ij} = a_{ji} \) for every \( i,j \). Show that this is a subspace of \( \mathcal{M}^{2,2} \) and find a basis and its dimension.
2.1) Given \( A = \begin{pmatrix} 3 & 1 & -2 \\ 4 & 0 & 2 \end{pmatrix} \) \( B = \begin{pmatrix} -2 & 0 & 1 \\ 6 & 4 & -1 \end{pmatrix} \), calculate \( C = 2A - 3B \).

**Solution.** We use the definition of sum introduced in the space of \( 2 \times 3 \) matrices:

\[
C = 2 \begin{pmatrix} 3 & 1 & -2 \\ 4 & 0 & 2 \end{pmatrix} - 3 \begin{pmatrix} -2 & 0 & 1 \\ 6 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 2 & -4 \\ 8 & 0 & 4 \end{pmatrix} + \begin{pmatrix} -18 & 0 & -3 \\ -10 & -12 & 7 \end{pmatrix} = \begin{pmatrix} 12 & 2 & -7 \\ -10 & -12 & 7 \end{pmatrix}.
\]

2.3) Determine if the vector \( \mathbf{v} = (2, -5, 3) \) in \( \mathbb{R}^3 \) is a linear combination of \( \mathbf{u}_1 = (1, -3, 2), \mathbf{u}_2 = (2, -4, -1) \) and \( \mathbf{u}_3 = (1, -5, 7) \).

**Solution.** Using the definition of linear combination of vectors in a vector space, we must prove that there exist real numbers \( a, b, c \) such that \( \mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3 \). Thus we look if the vector equation

\[
(2, -5, 3) = a(1, -3, 2) + b(2, -4, -1) + c(1, -5, 7)
\]

has solutions. Evaluating the sum on the right hand side we get

\[
(2, -5, 3) = (a + 2b + c, -3a - 4b - 5c, 2a - b + 7c)
\]

that leads to the system of linear equations

\[
\begin{align*}
a + 2b + c &= 3 \\
-3a - 4b - 5c &= -5 \\
2a - b + 7c &= 2.
\end{align*}
\]

We perform a Gaussian reduction and with operations \( R_2 + 3R_1 \) and \( R_3 - 2R_1 \) we transform the system into

\[
\begin{align*}
a + 2b + c &= 3 \\
2b - 2c &= 4 \\
-5b + 5c &= -4
\end{align*}
\]

a last row operation \( 2R_3 + 5R_2 \) leads to

\[
\begin{align*}
a + 2b + c &= 3 \\
2b - 2c &= 4 \\
0 &= 12
\end{align*}
\]

that is a system without solutions. We conclude that the vector \( \mathbf{v} \) cannot be written as a linear combination of \( \mathbf{u}_1, \mathbf{u}_2, \) and \( \mathbf{u}_3 \).

2.5) Determine if the matrix \( M = \begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix} \) is a linear combination of \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), \( B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \), \( C = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix} \).

**Solution.** We must determine if there exist real numbers \( a, b, c \) such that \( M = aA + bB + cC \), thus we must solve

\[
\begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + c \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}
\]

equivalent to

\[
\begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix} = a \begin{pmatrix} 1 & a \\ 1 & a \end{pmatrix} + b \begin{pmatrix} b & 2b \\ 3b & 4b \end{pmatrix} + c \begin{pmatrix} c & c \\ 4c & 5c \end{pmatrix}
\]

Performing the indicated operations

\[
\begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} a + b + c & a + 2b + c \\ a + 3b + 4c & a + 4b + 5c \end{pmatrix}
\]

that leads to the system of linear equations

\[
\begin{align*}
a + b + c &= 4 \\
a + 2b + c &= 7 \\
a + 3b + 4c &= 7 \\
a + 4b + 5c &= 9.
\end{align*}
\]
We perform a Gaussian reduction and with operations \( R_2 - R_1, R_3 - R_1 \) and \( R_4 - R_1 \) we transform the system into
\[
\begin{align*}
a + b + c &= 4 \\
b &= 3 \\
2b + 3c &= 3 \\
3b + 4c &= 5
\end{align*}
\]
From last two equations we deduce that \( c = -1 \), from the second \( b = 3 \), and, substituting into the first, \( a = 2 \). We conclude that \( M = 2A + 3B - C \).

**2.7** In \( M^2 \) show that \( A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, B = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} \) and \( C = \begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix} \) are Linearly Dependent.

**Solution.** By definition, \( n \) vectors in a linear space are linearly dependent if at least one of them can be written as a linear combination of the others, but this condition is not simple to verify. Thus we use a basic theorem: \( n \) vectors in a linear space are linearly dependent if and only if there exists a non trivial combination of the vectors that equals the null vector of the space. This condition is easier to verify. We must prove that there exist real numbers \( a, b, c \), not all equal to zero, such that \( aA + bB + cC = 0 \), where \( O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).

We solve
\[
a \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} + b \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} + c \begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]
We perform the sum on the left hand side:
\[
\begin{pmatrix} a & 2a \\ 3a & a \end{pmatrix} + \begin{pmatrix} 3b & -b \\ 2b & 2b \end{pmatrix} + \begin{pmatrix} c & -5c \\ -4c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]
This leads to the homogeneous system of linear equations:
\[
\begin{align*}
a + 3b + c &= 0 \\
2a - b - 5c &= 0 \\
3a + 2b - 4c &= 0 \\
a + 2b &= 0.
\end{align*}
\]
We perform a Gaussian reduction and with operations \( R_2 - 2R_1, R_3 - 3R_1 \) and \( R_4 - R_1 \) we transform the system into:
\[
\begin{align*}
a + 3b + c &= 0 \\
-7b - 7c &= 0 \\
-7b - 7c &= 0 \\
-b - c &= 0.
\end{align*}
\]
This system has infinitely many solutions, for any possible real value of \( c \), it is enough to choose \( b = -c \) and \( a = 2c \) to find a solution. To get a non trivial solution, we may choose \( c = 1 \), thus \( b = -1 \) and \( a = 2 \) and \( 2A - B + C = O \) implies that \( A, B, C \) are linearly dependent.

**2.9** Prove that \( v_1 = (2, -3, 1), v_2 = (3, -1, 5), v_3 = (1, -4, 3) \) are linearly independent.

**Solution.** We must prove that the only linear combination of \( v_1, v_2, v_3 \) that equals the zero vector of \( R^3 \) is the trivial linear combination. We prove that if \( a v_1 + b v_2 + c v_3 = 0 \) then \( a = b = c = 0 \), where \( O = (0, 0, 0) \). Suppose that \( a v_1 + b v_2 + c v_3 = 0 \), this means that
\[
a(2, -3, 1) + b(3, -1, 5) + c(1, -4, 3) = (0, 0, 0)
\]
that implies \( 2a + 3b + c, -3a - b - 4c, a + 5b + 3c = 0, 0, 0 \).

This is equivalent to the homogeneous system
\[
\begin{align*}
2a + 3b + c &= 0 \\
-3a - b - 4c &= 0 \\
a + 5b + 3c &= 0.
\end{align*}
\]
We perform a Gaussian reduction, with operations $R_3 \leftrightarrow R_1$, $R_1 - 2R_3$, $R_2 + 3R_3$ we get

\[
\begin{align*}
a + 5b + 3c &= 0 \\
-7b - c &= 0 \\
14b - c &= 0
\end{align*}
\sim
\begin{align*}
a + 5b + 3c &= 0 \\
-7b - c &= 0 \\
-3c &= 0
\end{align*}
\]

This system has a unique solution $a = b = c = 0$, thus the given vectors are linearly independent in $\mathbb{R}^3$.

**Observation.** If we solve the exercise after studying matrices and determinants, we may use a different approach. Three vectors in $\mathbb{R}^3$ are linearly independent if and only if the matrix formed by the three vectors as rows (or columns) has rank = 3, if and only if the determinant of such matrix is different from zero. Thus we may calculate $\det \begin{pmatrix} 2 & -3 & 1 \\ 3 & -1 & 5 \\ 1 & -4 & 3 \end{pmatrix} = -6 - 15 - 12 + 1 + 40 + 27 = 36 \neq 0$ and conclude that the given vectors are linearly independent.

2.11) For what values of $\lambda \in \mathbb{R}$ are the polynomials $p_1(t) = 2 + 5t + t^2$, $p_2(t) = 4 + 10t + 8t^2$, $p_3(t) = 6 + 10t + \lambda t^2$ linearly dependent?

**Solution.** Three polynomials in $\mathcal{P}$ are linearly dependent if there exists a non trivial linear combination of the polynomials that equals the zero polynomial. We will look for the possible values of $\lambda$ such that

\[
ap_1(t) + ap_2(t) + cp_3(t) = 0
\]

has a solution with $a, b, c$ not all equal to zero. The previous equation is expanded into

\[
a(2 + 5t + t^2) + b(4 + 10t + 8t^2) + c(6 + 10t + \lambda t^2) = 0
\]

and, after performing the operations on the left hand side

\[
(2a + 4b + 6c) + (5a + 10b + 10c)t + (a + 8b + \lambda c)t^2 = 0.
\]

This leads to the following homogeneous system and its Gaussian reduction

\[
\begin{align*}
2a + 4b + 6c &= 0 \\
5a + 10b + 10c &= 0 \\
a + 8b + \lambda c &= 0
\end{align*}
\sim
\begin{align*}
R_1/2
a + 2b + 3c &= 0 \\
R_2 - 5R_1/2
-15c &= 0 \\
R_3 - R_1/2
6b + (\lambda - 3)c &= 0.
\end{align*}
\]

The second equation implies that $c = 0$, substituting into the third equation we deduce $b = 0$, and finally substituting the obtained value into the first equation we get $a = 0$. The solution was deduced independently from $\lambda$, thus, for any possible value of $\lambda$ the given polynomials are linearly independent.

**Observation.** If we solve the exercise after studying matrices and determinants, we may use a different approach. Three vectors in $\mathcal{P}_2$ are linearly independent if and only if their coordinates with respect to a given basis of $\mathcal{P}_2$, seen as vectors of $\mathbb{R}^3$, are linearly independent, if and only if the corresponding matrix formed by the three vectors as rows (or columns) has rank = 3, if and only if the determinant of such matrix is different from zero. Thus we may consider the canonical basis of $\mathcal{P}_2$, $B = \{1, t, t^2\}$ and the coordinates of the given polynomials $[p_1]_B = (2, 5, 1)$, $[p_2]_B = (4, 10, 8)$, $[p_3]_B = (6, 10, \lambda)$ and evaluate

\[
\det \begin{pmatrix} 2 & 5 & 1 \\ 4 & 10 & 8 \\ 6 & 10 & \lambda \end{pmatrix} = 20\lambda + 40 + 240 - 60 - 160 - 60\lambda = 60 \neq 0.
\]

We conclude that the given polynomials are linearly independent for any possible value of $\lambda$.

2.13) Given $A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ -1 & -8 \\ -4 & 2 \end{pmatrix}$, $C = \begin{pmatrix} -3 & 4 \\ 1 & d \\ e & f \end{pmatrix}$, find $d, e, f$ so that $A$, $B$, $C$ are linearly dependent.

**Solution.** The given matrices are vectors of the linear space $\mathcal{M}^{3,2}$, they are linearly dependent if and only if we can find real numbers $\alpha, \beta, \gamma$, not all equal to zero, such that $\alpha A + \beta B + \gamma C = \mathbf{0}$ where $\mathbf{0}$ is the zero vector in $\mathcal{M}^{3,2}$. We try to find values for the parameters $d, e, f$ so that the equation $\alpha A + \beta B + \gamma C = \mathbf{0}$ is satisfied for some non trivial values of $\alpha, \beta, \gamma$. We evaluate $\alpha A + \beta B + \gamma C = \mathbf{0}$ and we get

\[
\alpha \begin{pmatrix} 1 & -2 \\ 0 & 3 \\ 1 & -1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ -1 & -8 \\ -4 & 2 \end{pmatrix} + \gamma \begin{pmatrix} -3 & 4 \\ 1 & d \\ e & f \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

if and only if
By definition, the span of a set of vectors is the set (we prove that is a linear space) formed by all possible linear combinations of the vectors. In this exercise we must prove that \( g \in \text{span}\{a, b, c\} \), show that \( g \) is linearly independent, i.e. we must find real values \( a, b, c \) such that \( g = ag_1 + bg_2 + cg_3 \). We pose the condition:

\[
5t - 7t^2 = (a + 7b + 3c) + (a - 8b - 2c)t + (-2a + 7b + c)t^2.
\]

This is equivalent to

\[
5t - 7t^2 = (a + 7b + 3c) + (a - 8b - 2c)t + (-2a + 7b + c)t^2.
\]

Two polynomial functions are equal if and only if they have same corresponding coefficients, thus we look for values of \( a, b, c \) that solve the system:

\[
\begin{align*}
  a + 7b + 3c &= 0 \\
  a - 8b - 2c &= 5 \\
  -2a + 7b + c &= -7.
\end{align*}
\]

We perform a Gaussian reduction with operations \( R_2 - R_1 \) and \( R_3 + 2R_1 \) to obtain

\[
\begin{align*}
  a + 7b + 3c &= 0 \\
  -15b - 5c &= 5 \\
  21b + 7c &= -7.
\end{align*}
\]

The system has infinitely many solutions: for any possible value \( \alpha \in \mathbb{R} \), \( a = 3 + 2\alpha \), \( b = \alpha \), \( c = -1 - 3\alpha \) is a solution of the system. Thus \( g \) can be written as a linear combination of \( g_1, g_2, g_3 \) in infinitely many ways, proving that \( g \in \text{span}\{g_1, g_2, g_3\} \).

2.17) Prove that in any linear space if \( x \) and \( y \) are linearly independent then \( x + y \) and \( x - y \) are two linearly independent vectors.

Solution. To prove that \( x + y \) and \( x - y \) are linearly independent, we will prove that the only linear combination of \( x + y \) and \( x - y \) that equals the zero vector is the trivial linear combination. Suppose that

\[
a(x + y) + b(x - y) = 0
\]

then

\[
a(x + y) + b(x - y) = (a + b)x + (a - b)y = 0.
\]

By hypothesis, \( x \) and \( y \) are linearly independent, i.e. the only linear combination of \( x \) and \( y \) that equals the zero vector is the trivial linear combination, thus the last equation implies the following system of conditions on \( a \) and \( b \)

\[
\begin{align*}
  a + b &= 0 \\
  a - b &= 0.
\end{align*}
\]
In order to prove the equality \( \text{span} \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} = \mathbb{R}^3 \), we must prove two inclusions. The first, \( \text{span} \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \subseteq \mathbb{R}^3 \), holds trivially. To prove the second, \( \mathbb{R}^3 \subseteq \text{span} \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \), we take an arbitrary vector \( \mathbf{v} = (x, y, z) \in \mathbb{R}^3 \) and prove that it can be generated as a linear combination of \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \), i.e., there exist real numbers \( a, b, c \) such that

\[
\mathbf{v} = a \mathbf{u}_1 + b \mathbf{u}_2 + c \mathbf{u}_3.
\]

Expanding the previous equality we get

\[
(x, y, z) = a(1, 1, 1) + b(1, 2, 3) + c(1, 5, 8) = (a + b + c, a + 2b + 5c, a + 3b + 8c).
\]

This leads to the following system:

\[
\begin{align*}
  a + b + c &= x \\
  a + 2b + 5c &= y \\
  a + 3b + 8c &= z
\end{align*}
\]

We start a Gaussian reduction with operations \( R_2 - R_1 \) and \( R_3 - R_1 \):

\[
\begin{align*}
  a + b + c &= x \\
  b + 4c &= y - x \\
  2b + 7c &= z - x \\
  R_2 - 2R_1 &\sim b + 4c = y - x \\
  R_3 - 2R_1 &\sim -c = z - x - 2y
\end{align*}
\]

Now we can conclude that, for any possible \( x, y, z \), the system has a unique solution for \( a, b, c \), thus the vector \( \mathbf{v} \) belongs to the span of \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \).

**Observation.** If we use the notion of basis, we can prove the statement just by proving that \( B = \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \) is a basis of \( \mathbb{R}^3 \). Let’s recall that, by definition, a basis of a linear space is a set of linearly independent vectors that span the space. A theorem states that a basis is a maximal set of linearly independent vectors, thus, knowing that the dimension of \( \mathbb{R}^3 \) is three, if we prove that \( B = \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \) is a linearly independent set, it follows that it must be a basis, and thus \( \text{span} \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} = \mathbb{R}^3 \). In order to prove that the three vectors in \( \mathbb{R}^3 \) are linearly independent, we then calculate the determinant of the matrix formed with the three vectors as rows (or columns), and because

\[
\det \begin{pmatrix}
  \mathbf{u}_1 \\
  \mathbf{u}_2 \\
  \mathbf{u}_3
\end{pmatrix} = \det \begin{pmatrix}
  1 & 1 & 1 \\
  1 & 2 & 3 \\
  1 & 5 & 8
\end{pmatrix} = -1 \neq 0
\]

we conclude that \( B = \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \) is a basis of \( \mathbb{R}^3 \) and \( \text{span} \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} = \mathbb{R}^3 \).

**2.21** Given \( p_1(t) = 1, p_2(t) = t - 3, p_3(t) = (t - 1)(t + 2) \), show that \( B = \{ p_1, p_2, p_3 \} \) is a basis of the space \( P_2(t) \) of all polynomial functions with degree \( \leq 2 \). Find the coordinates of \( p(t) = 2t^2 + t + 2 \) with respect to \( B \).

**Solution.** By definition, a basis of a linear space is a set of linearly independent vectors that span the space. A theorem states that a basis is a maximal set of linearly independent vectors, thus, knowing that the dimension of \( P_2(t) \) is three, if we prove that \( B = \{ p_1, p_2, p_3 \} \) is a linearly independent set, it follows that it must be a basis. Let’s prove directly that \( p_1, p_2, p_3 \) are linearly independent. Suppose that a linear combination of the three polynomials equals the zero polynomial,

\[
ap_1(t) + bp_2(t) + cp_3(t) = 0
\]

then we must prove that \( a = b = c = 0 \). Explicitly expressing the previous equality, we get

\[
a(1) + b(t - 3) + c(t - 1)(t + 2) = 0 \quad \text{that is} \quad (a - 3b - 2c) + (b + c)t + ct^2 = 0.
\]

The last equality leads to the linear system

\[
\begin{align*}
  a - 3b - 2c &= 0 \\
  b + c &= 0 \\
  c &= 0
\end{align*}
\]
that has the only trivial solution $a = b = c = 0$. Thus $B = \{p_1, p_2, p_3\}$ is a basis of the space $\mathcal{P}_2(t)$. To find the coordinates of $p(t) = 2t^2 + t + 2$ with respect to $B$, we look for real numbers $a, b, c$ such that

$$a p_1(t) + b p_2(t) + c p_3(t) = p(t).$$

This means

$$a(1) + b(t-3) + c(t-1)(t+2) = 2t^2 + t + 2 \quad \text{that is} \quad (a - 3b - 2c) + (b + c)t + ct^2 = 2t^2 + t + 2.$$

The last equality leads to the linear system

$$a - 3b - 2c = 2$$

$$b + c = 1$$

$$c = 2$$

that has the unique solution $a = 3, b = -1, c = 2$, thus $[p(t)]_B = (3,-1,2)$.

**Observation.** A faster way to prove that the three polynomials $p_1, p_2, p_3$ are linearly independent. Three polynomials in $\mathcal{P}_2(t)$ are linearly independent if and only if their coordinates with respect to a given basis of $\mathcal{P}_2$, seen as vectors of $\mathbb{R}^3$, are linearly independent, if and only if the corresponding matrix formed by the three vectors as rows (or columns) has rank $= 3$, if and only if the determinant of such matrix is different from zero. Thus we may consider the canonical basis of $\mathcal{P}_2$, $C = \{1, t, t^2\}$ and the coordinates of the given polynomials $[p_1]_C = (1,0,0), [p_2]_C = (-3,1,0), [p_3]_C = (-2,1,1)$ and evaluate $\det \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} = 1 \neq 0$.

We conclude that the given polynomials are linearly independent and thus they form a basis of $\mathcal{P}_2(t)$.

**2.23** Given $p_1 = 1 - x^2, p_2 = 1 + x, p_3 = -1 + x + x^3$, extend $\{p_1, p_2, p_3\}$ to a basis of $\mathcal{P}_3(t)$. (Use coordinates)

**Solution.** Given a linear space $L$ with finite dimension $n$, we can always consider the isomorphism between $L$ and $\mathbb{R}^n$ that associate to any vector of $L$ its coordinates with respect to a fixed ordered basis of $L$. Then the "linear" property that we wish to prove for a given set of vectors in $L$ can be proved for the corresponding vectors in $\mathbb{R}^n$ and vicerese. We will work with the coordinates of $p_1, p_2, p_3$ with respect to the standard basis $C$ of $\mathbb{R}^3, C = \{1, t, t^2, t^3\}$. Then $[p_1]_C = (1,0,-1,0), [p_2]_C = (1,1,0,0), [p_3]_C = (-1,1,0,1)$. Let us use the symbol $\sim$ to denote the coincidence of the corresponding spans. We have

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 1 \neq 0.$$

We see that adding the vector $v = (0,0,0,1)$ to the last set of three vectors, we will create a set of four linearly independent vectors of $\mathbb{R}^4$, thus a basis of $\mathbb{R}^4$. The polynomial $g = t^3$ is such that $[g]_C = v$, thus the set $\{p_1, p_2, p_3, g\}$ is a basis of $\mathcal{P}_3(t)$.

**Observation.** Using the notion of matrix and the fact that four vectors in $\mathbb{R}^4$ are linearly independent if and only if the rank of the matrix formed with the vectors as rows is regular: we can try to guess and add any row of coordinates to

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix}$$

so that the resulting $4 \times 4$ matrix has non zero determinant. For example, adding $(1,0,0,0)$ we get

$$\det \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -1 \neq 0$$

and this implies that also the polynomial $q(t) = 1$ (for which $[q]_C = (1,0,0,0)$) is such that $\{p_1, p_2, p_3, q\}$ is a basis of $\mathcal{P}_3(t)$.

**2.25** Given the linear space $\mathcal{P}(t)$ of all polynomials, determine if the following are linear subspaces
i) all polynomials with integer coefficients
ii) all polynomials with even powers of $t$ (i.e. containing only terms of the type $t^{2n}$ where $n=0,1,2...$)
iii) all polynomials with degree greater or equal to six.
Solution. By definition, a non empty subset $M$ of a linear space $L$ is a subspace of $L$ if:
1) given any two vectors $\mathbf{u}, \mathbf{v} \in M$, then $\mathbf{u} + \mathbf{v} \in M$,
2) given any vector $\mathbf{u} \in M$ and any real number $\alpha$, then $\alpha \cdot \mathbf{u} \in M$.

For the given sets of polynomials we have:
i) all polynomials with integer coefficients do not form a linear subspace of $\mathcal{P}(t)$. Indeed, condition 2) is not satisfied, it is enough to consider any polynomial with integer coefficient, for example $p(t) = t$, and the real number $\alpha = \sqrt{2}$, to see that the polynomial $\alpha \cdot p(t) = \sqrt{2}t$ is no more a polynomial with integer coefficient.

ii) all polynomials containing only terms of the type $t^{2n}$ where $n = 0, 1, 2...$ form a linear subspace of $\mathcal{P}(t)$. Indeed this subset of $\mathcal{P}(t)$ is not empty (for example $p(t) = t^2$ belongs to it) and 1) adding any two polynomial containing only terms with even powers of $t$ we create a new polynomial with only even powers of $t$;
2) multiplying any polynomial containing only terms with even power of $t$ by any non-zero number, we get a new polynomial containing exactly the same powers of $t$. While, if we multiply any polynomial by $0 \in \mathbb{R}$ we get the polynomial $z(t) = 0$ that belongs to the set by definition. We see how here it is of basic importance that in the definition of the subset the values of $n$ where specified as $n = 0, 1, 2...$, because if the value $n = 0$ was not included, the given subset would not be a subspace.

iii) all polynomials with degree greater or equal to six do not form a linear subspace of $\mathcal{P}(t)$. Indeed, both conditions are not satisfied. To see that condition 1) is not satisfied, let’s consider $p_1(t) = 5t - t^6$ and $p_2(t) = 1 - 3t^2 + t^6$, both polynomial have degree six, but $p_1(t) + p_2(t) = (5t - t^6) + (1 - 3t^2 + t^6) = 1 + 5t - 3t^2$
has degree three, less than six. For condition 2): if we multiply any polynomial with degree greater or equal to six by $0 \in \mathbb{R}$ we get the zero polynomial $z(t) = 0$, that is not in this subset of $\mathcal{P}(t)$ because it does not have a degree equal or greater than six.

2.27) In the linear space $\mathcal{F}$ of all real functions $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$, decide if the sets defined below form linear subspaces or not.
$M_1 = \{f \in \mathcal{F} : f(1) = 0\}$,
$M_2 = \{f \in \mathcal{F} : f(3) = 1\}$.

Solution. At the beginning of the definition of linear subspace. We use it to prove that $M_1 = \{f \in \mathcal{F} : f(1) = 0\}$ is a linear subspace of $\mathcal{F}$. $M_1$ is clearly not empty as the zero function $f(x) = 0$ belongs to it. In order to verify condition 1), let’s consider two functions $f_1, f_2$ belonging to $M_1$, for such functions $f_1(1) = f_2(1) = 0$, therefore for their sum we have
$$(f_1 + f_2)(1) = f_1(1) + f_2(1) = 0 + 0 = 0$$
thus $f_1 + f_2$ belongs to $M_1$. To verify condition 2), we consider $f \in \mathcal{F}$ and $\alpha \in \mathbb{R}$, then for $\alpha \cdot f$ we have
$$(\alpha \cdot f)(1) = \alpha \cdot f(1) = \alpha \cdot 0 = 0$$
thus $\alpha f$ belongs to $M_1$.

$M_2 = \{f \in \mathcal{F} : f(3) = 1\}$ is not a linear subspace of $\mathcal{F}$. Indeed, both conditions are not satisfied. Considering two functions $f_1, f_2$ belonging to $M_2$, for such functions $f_1(3) = f_2(3) = 1$, therefore for their sum we have
$$(f_1 + f_2)(3) = f_1(3) + f_2(3) = 1 + 1 = 2 \neq 1$$
thus $f_1 + f_2$ does not belong to $M_2$. This is sufficient to say that $M_2$ is not a linear subspace. Anyway, we may also observe that also condition 2) is not satisfied: considering $f \in M_2$ and $0 \in \mathbb{R}$ then $0 \cdot f = 0$ is the constant zero function, but this does not belong to $M_2$.

2.29) Given the following subsets of $\mathcal{P}_3(t)$ determine if they are linear subspaces of $\mathcal{P}_3(t)$ or not. In affirmative case, find the dimension and a basis for each subspace:
$L_1 = \{p \in \mathcal{P}_3 : p(0) = 0\}$
$L_2 = \{p \in \mathcal{P}_3 : p(-t) = p(t)\}$.

Solution. For the definition of linear subspace we refer to the beginning of the solution of exercise 25).
Let’s prove that $L_1 = \{p \in \mathcal{P}_3 : p(0) = 0\}$ is a linear subspace of $\mathcal{P}_3$. $L_1$ is not empty as it contains for example the zero polynomial. To verify condition 1), let’s consider two polynomials $p_1, p_2$ belonging to $L_1$, for such polynomials of degree at most three, $p_1(0) = p_2(0) = 0$, therefore for their sum we have
$$(p_1 + p_2)(0) = p_1(0) + p_2(0) = 0 + 0 = 0$$
thus \( p_1 + p_2 \) belongs to \( L_1 \). To verify condition 2), we consider \( p \in L_1 \) and \( \alpha \in \mathbb{R} \), then for \( \alpha \cdot p \) we have

\[
(\alpha \cdot p)(0) = \alpha \cdot p(0) = \alpha \cdot 0 = 0
\]

thus \( \alpha p \) belongs to \( L_1 \). Now we need to find a basis for \( L_1 \). Taken a general \( p(t) \in \mathcal{P}_3 \), \( p(t) = a + bt + ct^2 + dt^3 \) belongs to \( L_1 \) if and only if

\[
p(0) = a + b0 + c0 + d0 = a = 0
\]

thus any polynomial in \( L_1 \) is of the form \( p(t) = b(t) + c(t^2) + d(t^3) \) and it can be generated by the linearly independent polynomials \( t, t^2, t^3 \). We conclude that \( B = \{t, t^2, t^3\} \) is a basis of \( L_1 \) and \( \dim L_1 = 3 \).

Let’s now prove that \( L_2 = \{ p \in \mathcal{P}_3 : p(-t) = p(t) \} \) is also a linear subspace of \( \mathcal{P}_3 \). \( L_2 \) is not empty as it contains for example the zero polynomial. To verify condition 1), let’s consider two polynomials \( p_1, p_2 \) belonging to \( L_2 \), for such polynomials of degree at most three, \( p_1(-t) = p_1(t) \) and \( p_2(-t) = p_2(t) \), therefore for their sum we have

\[
(p_1 + p_2)(-t) = p_1(-t) + p_2(-t) = p_1(t) + p_2(t) = (p_1 + p_2)(t)
\]

thus \( p_1 + p_2 \) belongs to \( L_2 \). To verify condition 2), we consider \( p \in L_2 \) and \( \alpha \in \mathbb{R} \), then for \( \alpha \cdot p \) we have

\[
(\alpha \cdot p)(-t) = \alpha \cdot p(-t) = \alpha \cdot p(t) = (\alpha \cdot p)(t)
\]

thus \( \alpha p \) belongs to \( L_2 \). Now we need to find a basis for \( L_2 \). Taken a general \( p(t) \in \mathcal{P}_3 \), \( p(t) = a + bt + ct^2 + dt^3 \) belongs to \( L_2 \) if and only if \( p(-t) = p(t) \), that means

\[
a + b(-t) + c(-t)^2 + d(-t)^3 = a + bt + ct^2 + dt^3
\]

or

\[
a - bt + ct^2 - dt^3 = a + bt + ct^2 + dt^3.
\]

This condition implies that \( b = d = 0 \). Thus any polynomial in \( L_2 \) is of the form \( p(t) = a + ct^2 = a(1) + c(t^2) \) and it can be generated by the linearly independent polynomials \( 1, t^2 \). We conclude that \( B = \{1, t^2\} \) is a basis of \( L_2 \) and \( \dim L_2 = 2 \).

2.31) a) Prove that if \( M \) and \( N \) are subspaces of a vector space \( L \) then \( M \cap N \) is a subspace of \( L \).

b) Given \( M = \text{span}\{(1,2,1,0),(-1,1,1,1)\} \) and \( N = \text{span}\{(1,2,1,-2),(2,1,0,1)\} \) (subspaces of \( \mathbb{R}^4 \)), find the dimension and a basis for \( M \cap N \).

c) Consider the space \( U = \text{span}\{M \cup N\} \), find the dimension and a basis of \( U \).

Solution. For the definition of linear subspace we refer to the beginning of the solution of exercise 25).

a) We prove that the intersection of two subspaces of a linear space is again a subspace. Being \( M \) and \( N \) subspaces of a linear space \( L \), they both contain the zero vector of \( L \), thus the intersection \( M \cap N \) also contains the zero vector of \( L \) and it is non-empty. Suppose now that we have two vectors \( u, v \in M \cap N \), this implies that \( u, v \in M \) and \( u, v \in N \), thus \( u + v \in M \) because \( M \) is a subspace, and \( u + v \in N \) because \( N \) is a subspace. This implies that \( u + v \in M \cap N \) and condition 1) from the definition of linear subspace is satisfied.

To prove that also condition 2) is satisfied, let’s consider \( u \in M \cap N \) and \( \alpha \in \mathbb{R} \). Because \( u \in M \cap N \), \( u \in M \) and \( u \in N \), thus \( \alpha \cdot u \in M \) and \( \alpha \cdot u \in N \) because both \( M \) and \( N \) are linear subspace and they satisfied condition 2). This implies that \( \alpha \cdot u \in M \cap N \).

b) Given \( M = \text{span}\{(1,2,1,0),(-1,1,1,1)\} \) and \( N = \text{span}\{(1,2,1,-2),(2,1,0,1)\} \), \( M \cap N \) is composed by all vectors in \( \mathbb{R}^4 \) that can be written at the same time as linear combination of \( (1,2,1,0), (-1,1,1,1) \) and linear combination of \( (1,2,1,-2),(2,1,0,1) \). Given a vector \( v \in M \cap N \), there must be real numbers \( a, b, c, d \) such that

\[
v = (a,1,2,1,0) + b(-1,1,1,1) = c(1,2,1,-2) + d(2,1,0,1).
\]

This condition is equivalent to

\[
(a - b,2a + b,a + b,b) = (c + 2d,2c + d,c,-2c + d)
\]

and leads to the system

\[
\begin{align*}
    a - b &= c + 2d \\
    2a + b &= 2c + d \\
    a + b &= c \\
    b &= -2c + d.
\end{align*}
\]
We rewrite the system as

\begin{align*}
a - b - c - 2d &= 0 \\
2a + b - 2c - d &= 0 \\
a + b - c &= 0 \\
b + 2c - d &= 0
\end{align*}

and perform a Gaussian reduction. We start with operation \( R_d \)

We now see that, taking \( d = \alpha \) as a free variable, all possible solutions of the system can be written as \( a = 2\alpha \), \( b = -\alpha \), \( c = \alpha \), \( d = \alpha \), with \( \alpha \in \mathbb{R} \). Thus the vector \( \mathbf{v} \in M \cap N \) is such that

\[ \mathbf{v} = a(1,2,1,0) + b(-1,1,1,1) = 2\alpha(1,2,1,0) - \alpha(-1,1,1,1) = \alpha(3,3,1,-1) \]

or equivalently

\[ \mathbf{v} = c(1,2,1,-2) + d(2,1,0,1) = \alpha(1,2,1,-2) + \alpha(2,1,0,1) = \alpha(3,3,1,-1). \]

We conclude that \( M \cap N = \text{span}\{3,3,1,-1\} \), and \( \dim M \cap N = 1 \).

c) To find the dimension and a basis of \( U = \text{span}\{M \cup N\} \), we need to select a set of linearly independent vectors that generate the set

\[ U = \text{span}\{M \cup N\} = \text{span}\{(1,2,1,0), (-1,1,1,1), (1,2,1,-2), (2,1,0,1)\} \]

Let us use the symbol \( \sim \) to denote the coincidence of the corresponding spans. We have

\[
\begin{array}{cccc}
(1,2,1,0) & R_2 + R_1 & (1,2,1,0) & R_3/2 - R_2/3 \\
(-1,1,1,1) & \sim & (0,3,2,1) & \sim \\
(1,2,1,-2) & R_3 - R_1 & (0,0,0,2) & R_4 + R_2 \\
(2,1,0,1) & R_4 - 2R_3 & (0,-3,-2,1) & \end{array}
\]

Considering that the last two vectors are obviously dependent, we conclude that the first three span the given space:

\[ U = \text{span}\{M \cup N\} = \text{span}\{(1,2,1,0), (0,3,2,1), (0,0,0,1)\} \]

\[ B = \{(1,2,1,0), (0,3,2,1), (0,0,0,1)\} \] is a basis of \( U \), and \( \dim U = 3 \).

2.33) Consider \( M = \{p(t) \in \mathcal{P}^3(t) : p(t) \text{ is divisible by } (t - 1)\} \). Prove that \( M \) a linear subspace of \( \mathcal{P}^3 \), find a basis and dimension of \( M \).

**Solution.** For the definition of linear subspace we refer to the beginning of the solution of exercise 25).

The divisibility by a given polynomial means that the remainder after division is zero. Every polynomial \( p(t) \in M \) is either the zero polynomial \((0 \in M \neq \emptyset)\) or can be written in the form \( p(t) = (t - 1)q(t) \), where \( q(t) \) is a polynomial with degree at most two. Given two polynomials \( p_1, p_2 \in M \), we have \( p_1(t) = (t - 1)q_1(t) \) and \( p_2(t) = (t - 1)q_2(t) \), where \( q_1, q_2 \in \mathcal{P}^2(t) \), thus

\[ p_1(t) + p_2(t) = (t - 1)q_1(t) + (t - 1)q_2(t) = (t - 1)[q_1(t) + q_2(t)]. \]

This proves that \( p_1(t) + p_2(t) \) is divisible by \( (t - 1) \). Analogously, for every \( \alpha \in \mathbb{R} \) and \( p(t) \in M \), there exists a \( q \in \mathcal{P}^2(t) \) so that \( p(t) = (t - 1)q(t) \), and we have

\[ \alpha \cdot p(t) = \alpha \cdot (t - 1)q(t) = (t - 1)[\alpha \cdot q(t)]. \]

This proves that \( \alpha \cdot p(t) \) is divisible by \( (t - 1) \). Thus \( M \) is a linear subspace of \( \mathcal{P}^3(t) \).

Suppose now that \( p(t) = at^3 + bt^2 + ct + d \) is a polynomial in \( M \), the fact that \( p(t) \) is divisible by \( (t - 1) \) implies that the remainder in the following division is zero:
Consider polynomials, this proves that the set 
\[ p(t) = at^3 + bt^2 + ct + d : (t - 1) = at^2 + (a + b)t + (a + b + c) \]
\[ - (a^3 - at^2) \]
\[ (a + b)t^2 + t + d \]
\[ - ((a + b)t^2 - (a + b)t) \]
\[ (a + b + c)t + d \]
\[ - ((a + b + c)t - (a + b + c)) \]
\[ (a + b + c + d) = 0 \]

We see that the coefficients of \( p(t) \in M \) must be such that \((a + b + c + d) = 0\). This means that a polynomial \( p(t) \) belongs to \( M \) if and only if there are coefficients \( a, b, c \) such that
\[ p(t) = at^3 + bt^2 + ct - a - b - c = a(t^3 - 1) + b(t^2 - 1) + c(t - 1). \]

This proves that the set \( B = \{t^3 - 1, t^2 - 1, t - 1\} \) spans \( M \) and, because \( B \) is made of linearly independent polynomials, \( B \) is a basis of \( M \), and \( \dim M = 3 \).

2.35 Consider \( Q = \{p(x) \in \mathcal{P}^4(x) : p(x) = x^4 \cdot p(x) \} \) for any \( x \in \mathbb{R} \neq 0 \). Prove that \( Q \) a linear subspace of \( \mathcal{P}^4 \), find a basis and dimension of \( Q \).

**Solution.** For the definition of linear subspace we refer to the beginning of the solution of exercise 25). To see that \( Q \) is a non-empty set we may observe that the zero polynomial obviously satisfied the condition in the definition of \( Q \), but also for example the polynomial \( p(x) = x^2 \) is such that \( p(x) = x^4 \cdot p(x) \) is satisfied. Thus also \( x^2 \in Q \). In general, given \( p_1, p_2 \in Q \), for every \( x \in \mathbb{R} \), we have \( p_1(x) = x^4 p_1(x) \) and \( p_2(x) = x^4 p_2(x) \), thus:
\[ p_1(x) + p_2(x) = x^4 p_1(x) + x^4 p_2(x) = x^4 [p_1(x) + p_2(x)] = x^4 (p_1 + p_2) \]

that means that \( p_1 + p_2 \in Q \). Moreover for any \( \alpha \in \mathbb{R} \) and \( p \in Q \), we have
\[ \alpha \cdot p(x) = \alpha \cdot x^4 p(x) = x^4 (\alpha \cdot p(x)) \]

thus \( \alpha \cdot p \in Q \), and this conclude the proof that \( Q \) is a linear subspace of \( \mathcal{P}^4(x) \).

Suppose now that \( p(x) = ax^4 + bx^3 + cx^2 + dx + e \) is a polynomial in \( Q \), the fact that \( p(x) = x^4 \cdot p\left(\frac{1}{x}\right) \) is equivalent to
\[ ax^4 + bx^3 + cx^2 + dx + e = x^4 \left(a \frac{1}{x^4} + b \frac{1}{x^3} + c \frac{1}{x^2} + d \frac{1}{x} + e\right) = a + bx + cx^2 + dx^3 + ex^4. \]

This condition is satisfied only if \( a = e \) and \( b = d \), thus \( p(x) \in Q \subset \mathcal{P}^4 \) if and only if there exist coefficients \( a, b, c \in \mathbb{R} \) such that
\[ p(x) = ax^4 + bx^3 + cx^2 + bx + a = a(x^4 + 1) + b(x^3 + x) + c(x^2). \]

This proves that the set \( B = \{x^4 + 1, x^3 + x, x^2\} \) is a basis of \( Q \), and \( \dim Q = 3 \).
3.1) Given \( A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix} \), evaluate: \( A^T \), \( A^2 \), rank \( A \).

3.2) Given \( B = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix} \), evaluate: \( B^T \), \( B \cdot B^T \), rank \( B \).

3.3) a) Given a matrix \( A \in M^{n,n} \), prove that the set \( L_A = \{ B \in M^{n,n} : BA = AB \} \) (the set of all matrices that commute with \( A \)) is a linear subspace of \( M^{n,n} \).

b) For \( A = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \) find basis and dimension of \( L_A \).

3.4) Find all matrices \( B \) such that \( B \cdot A = A \cdot B \) where \( A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \).

3.5) Solve the equation \( A + 3X = BC \), where \( A = \begin{pmatrix} 1 & 5 \\ -3 & 2 \end{pmatrix} \), \( B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \) and \( C = \begin{pmatrix} -2 & 6 \\ 0 & 4 \end{pmatrix} \).

3.6) Solve the following system:

\[
\begin{align*}
5X + 2Y &= 16A - B \\
8X - 5Y &= A - 18B
\end{align*}
\]

where \( A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{pmatrix} \), \( B = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \), \( X, Y \in M^{2,3} \).

3.7) Solve \( AX = B \), where \( A = \begin{pmatrix} 2 & -1 & -1 \\ -2 & -1 & 1 \end{pmatrix} \), \( B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

3.8) Find (all possible) \( X \in M^{2,2} \) such that \( AX = X \) where \( A = \begin{pmatrix} 2 & 4 \\ -3 & -11 \end{pmatrix} \).

3.9) Find \( \lambda \in R \) so that \( A = \begin{pmatrix} 1 & -2 & 3 \\ 1 & \lambda & -1 \\ 1 & 1 & -\lambda \end{pmatrix} \) has rank equal to two. For \( \lambda = 1 \), verify that rank \( A = \text{rank} A^T \).

3.10) Find all \( \lambda \in R \) such that \( A = \begin{pmatrix} 1 & -\lambda & 0 & 1 \\ 2 & 2 + \lambda & 0 \\ 0 & 2 & 1 \end{pmatrix} \) has rank \( A = 2 \).

3.11) Find the determinant of \( A \) a) reducing it to triangular form, b) using the theorem on expansion of determinants: \( A = \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 4 \\ 3 & 2 & 0 & -5 \\ 4 & 3 & -5 & 0 \end{pmatrix} \). Then compute \( \text{det} A^{-1} \) and \( \text{det} A^2 \).

3.12) Find the determinant of \( A \) a) reducing it to triangular form, b) using the theorem on expansion of determinants: \( A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 1 & 1 \end{pmatrix} \). Then compute \( \text{det} A^{-1} \) and \( \text{det} A^{-2} \).

3.13) Use the determinant to find the value of rank \( A \) for any possible \( \alpha \in R \), where \( A = \begin{pmatrix} 16 & 0 & 4\alpha \\ 0 & 7 & 4\alpha \\ -\alpha & 0 & -1 \end{pmatrix} \).
3.14) Use the determinant of a matrix to find the values of \( \lambda \in \mathbb{R} \) for which the vectors \( v_1 = (2, 5, 1) \), \( v_2 = (4, 10, 8) \), \( v_3 = (6, 10, \lambda) \) are linearly dependent.

3.15) For each of the following matrices find the inverse first using Gauss elimination method, then using the formula \( A^{-1} = \frac{1}{\det A} \text{adj} A \), where \( \det A \) is the determinant of \( A \) and \( \text{adj} A \) is the classical adjoint matrix.

\[
A = \begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & 1 & 2 & -1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 2 & 3 \\
-1 & 1 & 2 \\
2 & 1 & -1
\end{pmatrix}.
\]

3.16) For each of the following matrices find the inverse first using Gauss elimination method, then using the formula \( A^{-1} = \frac{1}{\det A} \text{adj} A \), where \( \det A \) is the determinant of \( A \) and \( \text{adj} A \) is the classical adjoint matrix.

\[
A = \begin{pmatrix}
1 & 0 & 1 \\
1 & -1 & 1 \\
2 & 1 & 1
\end{pmatrix}, \quad P = \begin{pmatrix}
1 & 1 & 0 \\
1 & -4 & 2 \\
1 & -1 & 1
\end{pmatrix}.
\]

3.17) a) Show that if \( A \in M^{3,3} \) is upper triangular, i.e. \( A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & 0 & a_{33}
\end{pmatrix} \) then \( A^{-1} \) is upper triangular, i.e.

\[
A^{-1} = \begin{pmatrix}
b_{11} & b_{12} & b_{13} \\
0 & b_{22} & b_{23} \\
0 & 0 & b_{33}
\end{pmatrix}.
\]

b) Find inverse of: \( B = \begin{pmatrix}
1 & 0 & 0 \\
3 & 1 & 0 \\
0 & 3 & 1
\end{pmatrix} \).

3.18) Given the matrix \( A = \begin{pmatrix}
1 & a & a \\
a & b & b \\
a & c & c
\end{pmatrix} \), verify that \( \det A = (c-b)(b-a)(c-a) \) and find \( A^{-1} \).

3.19) Given \( A = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \). Find \( \text{adj} A \). When does \( A = \text{adj} A \)? Show that \( \text{adj}(\text{adj} A) = A \). Find \( A^{-1} \).

3.20) Given \( A = \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 2 & 1
\end{pmatrix} \). Find \( \det A \), \( \text{adj} A \) and \( A^{-1} \).

3.21) Given the matrix \( A = \begin{pmatrix}
1 & -2 & 1 \\
0 & 2 & 0 \\
-1 & 0 & 1
\end{pmatrix} \) and the vector \( b = \begin{pmatrix}
3 \\
0 \\
1
\end{pmatrix} \), find the inverse of the matrix \( A \) to solve the system \( Ax = b \).

3.22) Using an inverse matrix solve the matrix equation \( AX = B \) where

\[
A = \begin{pmatrix}
1 & -1 & 0 \\
-1 & 1 & -1 \\
1 & 0 & 1
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
3 & 4 & 5
\end{pmatrix}.
\]

3.23) Determine for what value of \( a \in \mathbb{R} \) the following matrix is regular and for those value find the inverse matrix:

\[
A = \begin{pmatrix}
a + 1 & 1 & -a \\
1 & a & 1 \\
0 & -1 & a
\end{pmatrix}.
\]

3.24) Given \( A = \begin{pmatrix}
1 & 2 \\
-3 & 1 \\
3 & -1
\end{pmatrix} \), \( B = \begin{pmatrix}
1 & 0 \\
3 & 2 \\
4 & 5
\end{pmatrix} \), find \( (B^TA)^{-1} \).

3.25) Use an inverse matrix to solve the matrix equation \( XA = (X + I)B \) where \( I \) is the identity matrix,

\[
A = \begin{pmatrix}
6 & 9 & 1 \\
4 & 2 & 4 \\
-1 & 0 & -1
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
5 & 7 & 1 \\
2 & 0 & 3 \\
-1 & 1 & 2
\end{pmatrix}.
\]
3.26) Use an inverse matrix to solve the matrix equation $AX = B - X$ where

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 4 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 7 & 1 \\ 2 & 0 & 3 \\ -1 & 1 & 2 \end{pmatrix}.$$
3.1) Given \( A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix} \), evaluate: \( A^T \), \( A^2 \), \( \text{rank } A \).

Solution. By definition the transpose of a matrix is a new matrix whose rows are the columns of the original. In our case, \( A \) is a \( 4 \times 4 \) matrix, i.e. \( A \in \mathcal{M}^{4,4} \), thus \( A^T \in \mathcal{M}^{4,4} \) and

\[
A^T = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix}.
\]

In order to evaluate \( A^2 = A \cdot A \) we need to perform a product of matrices. We recall that, in general, given matrices \( A = (a_{ik}) \in \mathcal{M}^{m,p} \), and \( B = (b_{kj}) \in \mathcal{M}^{p,n} \), the product \( A \cdot B = C = (c_{ij}) \) is a matrix in the space \( \mathcal{M}^{m,n} \), and

\[
c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj} \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n.
\]

This means that, in the \( ij \) position, the product matrix \( C \) has the result of the "scalar product" of the \( i \)-th row of \( A \) with the \( j \)-th column of \( B \) seen as vectors of \( \mathbb{R}^p \).

In our case \( n = m = p = 4 \), thus \( A \cdot A \) is a matrix in the space \( \mathcal{M}^{4,4} \). Now we evaluate:

\[
A^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1+0+2-1 & 0+0+1+1 & 1+0+1+0 & 1+0+1-1 \\ 0+0-2-2 & 0+4-1+2 & 0-2-1+0 & 0+4-1-2 \\ 2+0+2-1 & 0+2+1+1 & 2-1+1+0 & 2+2+1-1 \\ -1+0+0+1 & 0+2+0-1 & -1-1+0+0 & -1+2+0+1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 & 1 \\ 0 & 5 & -3 & 1 \\ 3 & 4 & 2 & 4 \\ 0 & 1 & -2 & 2 \end{pmatrix}
\]

By definition, the rank of a \( m \times n \) matrix is the dimension of the linear space spanned by the \( m \) rows of the matrix, seen as vectors in \( \mathbb{R}^n \). Equivalently, we may say that \( \text{rank } A \) is the maximal number of linearly independent rows of \( A \). Due to the fact (Theorem) that a Gaussian elimination does not change the rank, we use a Gaussian reduction:

\[
A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix} \quad R_3 - R_1 \sim R_4 - R_1 \sim \quad 2R_3 + R_2 \sim R_4 - R_2 \sim \quad R_4 - 2R_3 \sim \quad \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -7 \end{pmatrix}
\]

This implies that \( \text{rank } A = 4 \).

3.3) a) Given a matrix \( A \in \mathcal{M}^{n,n} \), prove that the set \( L_A = \{ B \in \mathcal{M}^{n,n} : BA = AB \} \) (the set of all matrices that commute with \( A \)) is a linear subspace of \( \mathcal{M}^{n,n} \).

b) For \( A = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \) find basis and dimension of \( L_A \).

Solution. a) For the definition of linear subspace we refer to the beginning of the solution of exercise 25).

Suppose that a matrix \( A \in \mathcal{M}^{n,n} \) is given, then the set \( L_A = \{ B \in \mathcal{M}^{n,n} : BA = AB \} \) contains the zero matrix because \( OA = AO = O \), thus \( L_A \neq \emptyset \). Suppose now that two matrices \( B_1, B_2 \in L_A \) are given, then \( B_1 A = AB_1 \) and \( B_2 A = AB_2 \), thus for \( B_1 + B_2 \) we have

\[
(B_1 + B_2)A = B_1A + B_2A = AB_1 + AB_2 = A(B_1 + B_2)
\]
due to the ("left" and "right") distributivity property of the sum with respect to the product of matrices. This proves that $B_1 + B_2 \in L_A$. In a similar way, using properties of matrix product, for any $B \in L_A$ and $\alpha \in \mathbb{R}$ we have 

$$(\alpha \cdot B)A = \alpha(AB) = A(\alpha B)$$

thus $\alpha \cdot B \in L_A$, and this concludes the proof that $L_A$ is a subspace of $M^{n,n}$.

b) Given $A = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$, the subspace $L_A$ is formed by all matrices $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a,b,c,d \in \mathbb{R}$, such that

$$(a \ b) \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} (a \ b).$$

Evaluating the indicated products, we have

$$(3a + b \ a) = (3a + c \ 3b + d).$$

This matrix equation is equivalent to the following system

$$3a + 2b = 3a + c$$
$$a = 3b + d$$
$$3c + 2d = 2a$$
$$c = 2b.$$ 

Eliminating the first equation, that is obviously equivalent to the last, we rewrite the system and use a Gaussian reduction:

$$\begin{array}{cccc}
a - 3b - d & = & 0 & \\
-2a + 3c + 2d & = & 0 & 2R_1 + R_2 \\
-2b + c & = & 0 & \sim
\end{array}$$

$$\begin{array}{cccc}
a - 3b - d & = & 0 & \\
-6b + 3c & = & 0 & 3R_3 - R_2 \\
-2b + c & = & 0 & \sim
\end{array}$$

We choose as free variables $b$ and $d$, and write all solutions of the given system as $a = 3\alpha + \beta$, $b = \alpha$, $d = \beta$ and $c = 2\alpha$, where $\alpha, \beta \in \mathbb{R}$. Then, any matrix $B \in L_A$ can be written as

$$\begin{pmatrix} 3a + \beta \\ 2\alpha \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix}$$

thus $L_A = \text{span}\{ \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix} \}$, and $\dim L_A = 2$.

3.5) Solve the equation $A + 3X = BC$, where $A = \begin{pmatrix} 1 & 5 \\ -3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 6 \\ 0 & 4 \end{pmatrix}$.

Solution. Using properties of matrix operations, we get $3X = BC - A$ and thus $X = \frac{1}{3}[BC - A]$. We evaluate the indicated operations:

$$X = \frac{1}{3}[BC - A] = \frac{1}{3}\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -2 & 6 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ -3 & 2 \end{pmatrix}\right]$$

$$= \frac{1}{3}\left[\begin{pmatrix} -2 & 14 \\ -6 & 34 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ -3 & 2 \end{pmatrix}\right] = \frac{1}{3}\left[\begin{pmatrix} -3 & 9 \\ -3 & 32 \end{pmatrix}\right] = \begin{pmatrix} -1 & 3 \\ -1 & 8 \end{pmatrix}.$$ 

3.7) Solve $AX = B$, where $A = \begin{pmatrix} 2 & -1 & -1 \\ -2 & -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution. We observe that $A \in M^{2,3}$ and $B \in M^{2,2}$. From this we deduce that $X$ must be a matrix in the space $M^{3,2}$.

Suppose that $X = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$, we are looking for $a,b,c,d,e,f \in \mathbb{R}$ so that

$$\begin{pmatrix} 2 & -1 & -1 \\ -2 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
After performing the product on the right hand side, we get
\[
\begin{pmatrix}
2a - c & c - e & 2b - d - f \\
-2a - c + e & -2b - d + f
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]
This is equivalent to the system of linear equations:
\[
\begin{align*}
2a - c - e &= 1 \\
2b - d - f &= 0 \\
-2a - c + e &= 0 \\
-2b - d + f &= 1.
\end{align*}
\]
With operations $R_2 \leftrightarrow R_3$, $R_3 + R_1$, $R_4 + R_2$, we start a Gaussian reduction and get the equivalent system:
\[
\begin{align*}
2a - c - e &= 1 \\
-2c &= 1 \\
2b - d - f &= 0 \\
-2d &= 1.
\end{align*}
\]
The system has infinitely many solutions: $c = d = -\frac{1}{2}$, $e = 2a - \frac{1}{2}$, $f = 2b + \frac{1}{2}$, for any possible $a, b \in \mathbb{R}$. We conclude that for any possible $a, b \in \mathbb{R}$ a matrix $X$ of the form
\[
X = \begin{pmatrix}
a & b \\
-\frac{1}{2} & -\frac{1}{2} \\
2a - \frac{1}{2} & 2b + \frac{1}{2}
\end{pmatrix}
\]
solves the given equation.

**3.9** Find $\lambda \in \mathbb{R}$ so that $A = \begin{pmatrix} 1 & -2 & 3 \\ 1 & \lambda & -1 \\ 1 & 1 & 1 - \lambda \end{pmatrix}$ has rank equal to two. For $\lambda = 1$, verify that $\text{rank } A = \text{rank } A^T$.

**Solution.** We use the Theorem that guarantees that Gaussian elimination does not change the rank of a matrix, and the usual symbol $\sim$ to denote the coincidence of rank, then we have
\[
\begin{pmatrix}
1 & -2 & 3 \\ 1 & \lambda & -1 \\ 1 & 1 & 1 - \lambda
\end{pmatrix}
\sim
\begin{pmatrix}
R_2 - R_4 & 1 & -2 & 3 \\ 0 & \lambda + 2 & -4 \\ 0 & 3 & -\lambda - 3
\end{pmatrix}
\]
If $\lambda = -2$, the last matrix has rank equal to three. Supposing that $\lambda \neq -2$, we continue the Gaussian elimination with operation $3R_2 - (\lambda + 2)R_3$ and we get the matrix
\[
\begin{pmatrix}
1 & -2 & 3 \\ 0 & \lambda + 2 & -4 \\ 0 & 0 & \lambda^2 + 5\lambda - 6
\end{pmatrix}.
\]
Since $\lambda^2 + 5\lambda - 6 = 0$ for $\lambda = -6$ and $\lambda = 1$, we conclude that if $\lambda \neq 1, \lambda \neq -6$, then $\text{rank } A = 3$, and if $\lambda = 1$ or $\lambda = -6$, then $\text{rank } A = 2$.

For $\lambda = 1$ the given matrix is $A = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$. Its rank is two (as we have already evaluated) obviously due to the fact that the second and third row are equal, while the first and second are independent. To find $\text{rank } A^T$, we perform a Gaussian reduction on $A^T$:
\[
A^T = \begin{pmatrix}
1 & 1 & 1 \\ -2 & 1 & 1 \\ 3 & -1 & -1
\end{pmatrix}
\sim
\begin{pmatrix}
R_2 + 2R_1 & 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & -4 & -4
\end{pmatrix}
\sim
\begin{pmatrix}
R_2/3 & 1 & 1 \\ R_3/4 + R_2/3 & 0 & 1 & 1 \\ 0 & 0 & 0
\end{pmatrix}.
\]
Thus $\text{rank } A^T = 2 = \text{rank } A$.

**3.11** Find the determinant of $A$ a) reducing it to triangular form, b) using the theorem on expansion of determinants: $A = \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 4 \\ 3 & 2 & 0 & -5 \\ 4 & 3 & -5 & 0 \end{pmatrix}$. Then compute $\text{det } A^{-1}$ and $\text{det } A^2$. 

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Solution. a) We recall that the determinant of a triangular matrix is equal to the product of the components in the diagonal. Moreover, if a matrix $B$ is obtained from a matrix $A$ using one of the following operations: i) exchanging two rows, 
ii) multiplying a row by a nonzero scalar $\lambda \in \mathbb{R}$, 
iii) substituting a row with the sum of that row with a linear combination of the others, then the new matrix is such that 
i) $\det B = -\det A$, 
ii) $\det B = \lambda \det A$, 
iii) $\det B = \det A$.
Keeping this in mind, we proceed with the elimination method:

$$
\begin{align*}
\det \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 4 \\ 3 & 2 & 0 & -5 \\ 4 & 3 & -5 & 0 \end{pmatrix} & = R_3 - 3R_1 = \det \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 4 \\ 3 & 2 & 0 & -5 \\ 4 & 3 & -5 & 0 \end{pmatrix} \\
\det \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 6 & -11 \\ 0 & 0 & 4 & -8 \end{pmatrix} & = R_4 - 4R_1 = -24 \det \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 6 & -11 \end{pmatrix} \\
\det \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 6 & -11 \end{pmatrix} & = 6 \det \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{pmatrix} = 24 \det \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{pmatrix} = 24 + 15 - 10 + 0 = 29
\end{align*}
$$

from which we deduce that $\det A = -24$.

b) We now use the formula $\det A = \sum_{j=1}^{n} a_{ij} A_{ij}$, where $A_{ij}$ is the $ij$-th cofactor of $A$, and evaluate the determinant of $A$ along the second row, fixing $i = 2$:

$$
\det A = 0 + 1(-1)^{2+2} \det \begin{pmatrix} 1 & -2 & 3 \\ 3 & 0 & -5 \\ 4 & -5 & 0 \end{pmatrix} + (-3)(-1)^{2+3} \det \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & -5 \\ 4 & 3 & 0 \end{pmatrix} + 4(-1)^{2+4} \det \begin{pmatrix} 1 & 0 & -2 \\ 3 & 2 & 0 \\ 4 & 3 & -5 \end{pmatrix}.
$$

To evaluate the determinant of the $3 \times 3$ matrices indicated, we now decide to use the formula (Sarrus’rule):

$$
\det \begin{pmatrix} a & b & c \\ d & e & f \\ r & s & t \end{pmatrix} = aet + bfr + cds - rec - sf a - dbt.
$$

Therefore:

$$
\det A = 0 + 40 - 45 - 0 - 25 + 0 + 3(0 + 27 + 24 + 15 - 0) + 4(-10 + 0 - 18 + 16 - 0 - 0) = -30 + 54 - 48 = -24.
$$

Knowing that in general $\det A^{-1} = \frac{1}{\det A}$ and $\det A^2 = (\det A \cdot A) = (\det A)^2$, we conclude that in our case $\det A^{-1} = \frac{1}{-24}$ and $\det A^2 = 24^2$.

3.13) Use the determinant to find the value of rank $A$ for any possible $\alpha \in \mathbb{R}$, where $A = \begin{pmatrix} 16 & 0 & 4\alpha \\ 0 & 7 & 4\alpha \\ -\alpha & 0 & -1 \end{pmatrix}$.

Solution. For a square matrix $A \in \mathcal{M}^{n,n}$ we have $\det A \neq 0 \Leftrightarrow \text{rank } A = n \Leftrightarrow$ the rows are linearly independent as vectors in $\mathbb{R}^n$. We start the discussion on the possible value of rank $A$ for the given matrix evaluating its determinant (using Sarrus’rule):

$$
\det A = \det \begin{pmatrix} 16 & 0 & 4\alpha \\ 0 & 7 & 4\alpha \\ -\alpha & 0 & -1 \end{pmatrix} = -16 \cdot 7 + 7 \cdot 4\alpha^2 = 7 \cdot 4(\alpha^2 - 4).
$$

Thus, if $\alpha \neq 2, \alpha \neq -2$, then $A$ is regular and $\text{rank } A = 3$. For the values of $\alpha$ that make the determinant equal to zero, to determine the exact value of rank $A$ (it could be 2 or 1), in general we substitute the value of $\alpha$ into the matrix and perform a Gauss reduction. But in our case we may observe that the first and second row of $A$ are linearly independent for any possible value of $\alpha$, thus, if $\alpha = 2$ or $\alpha = -2$, then $\text{rank } A = 2$.

3.15) For each of the following matrices find the inverse first using Gauss elimination method, then using the formula $A^{-1} = \frac{1}{\det A} \text{adj } A$, where $\text{det } A$ is the determinant of $A$ and $\text{adj } A$ is the classical adjoint matrix.
a) \[ A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{b) } B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix}. \]

Solution. a) We will write \( A \) and \( I \) (the identity matrix) one beside the other and carry on a Gaussian elimination on \( A \), simultaneously performing the same operations on \( I \). The final stage is reached when \( A \) is reduced to the identity matrix \( I \), then \( I \) is reduced to \( A^{-1} \).

\[
A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \left( \begin{array}{c} R_4 - R_3 \\ R_1 - R_2 \\ R_3 + R_4 \\ R_2 - 2R_3 \\ R_3 + R_4/ -2 \end{array} \right)
\]

\[
A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad \left( \begin{array}{c} R_2/2 - R_3 \\ R_3 + R_4 \end{array} \right)
\]

\[
A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1/2 & -5/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix} = A^{-1}
\]

We now evaluate the inverse of \( A \) using the formula \( A^{-1} = \frac{1}{\det A} \text{adj} A \). We start with calculating the determinant of \( A \) along the fourth row that contains only one nonzero entry:

\[
\det A = 1(-1)^{3+4} \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} = -2.
\]

The matrix \( \text{adj} A \) is the adjugate, or classical adjoint of the square matrix \( A \) and it is defined as the transpose of its cofactor matrix. We will thus evaluate all cofactors of the given matrix.

\[
A_{11} = (-1)^{1+1} \cdot \det \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = -2
\]

\[
A_{12} = (-1)^{1+2} \cdot \det \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} = 0
\]

\[
A_{13} = (-1)^{1+3} \cdot \det \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 0
\]

\[
A_{14} = (-1)^{1+4} \cdot \det \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 0
\]

\[
A_{21} = (-1)^{2+1} \cdot \det \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 2
\]

\[
A_{22} = (-1)^{2+2} \cdot \det \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = -2
\]
a Gauss reduction:

b) For the given matrix $A$, we start with finding $B^{-1}$ with a Gauss reduction:

$$A_{23} = (-1)^{2+3} \cdot \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$A_{24} = (-1)^{2+4} \cdot \det \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

$$A_{31} = (-1)^{3+1} \cdot \det \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} = 0$$

$$A_{32} = (-1)^{3+2} \cdot \det \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -1$$

$$A_{33} = (-1)^{3+3} \cdot \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$A_{34} = (-1)^{3+4} \cdot \det \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = -1$$

$$A_{41} = (-1)^{4+1} \cdot \det \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} = 0$$

$$A_{42} = (-1)^{4+2} \cdot \det \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} = 5$$

$$A_{43} = (-1)^{4+3} \cdot \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} = -2$$

$$A_{44} = (-1)^{4+4} \cdot \det \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

Now we can write $A^{-1} = -\frac{1}{2} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 5 & -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1/2 & -5/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix}^T$.

b) For the given matrix $B$ we follow the same approach as for the matrix $A$, we start with finding $B^{-1}$ with a Gauss reduction:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_2 + R_1 \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & -3 & -7 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} R_3 + R_2 \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/3 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1/3 & 0 \\ -1 & 1 & 1 \end{pmatrix} R_3/2 \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1/2 & 3/2 & 3/2 \\ -1/2 & 7/6 & 5/6 \\ 1/2 & -1/2 & -1/2 \end{pmatrix} R_1 - 2R_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & -5/6 & -1/6 \\ -1/2 & 7/6 & 5/6 \\ 1/2 & -1/2 & -1/2 \end{pmatrix} = B^{-1}$. 

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We now evaluate the inverse of $B$ using the formula $B^{-1} = \frac{1}{\det B} \text{adj} B$. We start by calculating the determinant of $B$ using Sarrus’ rule.

$$\det B = \det \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix} = -1 + 8 - 3 - 6 - 2 - 2 = -6.$$ 

We now proceed with the evaluation of $\text{adj} B$, calculating all cofactors of $B$:

\[
\begin{align*}
B_{11} &= (-1)^{1+1} \cdot \det \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = -3 \\
B_{12} &= (-1)^{1+2} \cdot \det \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} = 3 \\
B_{13} &= (-1)^{1+3} \cdot \det \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} = -3 \\
B_{21} &= (-1)^{2+1} \cdot \det \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} = 5 \\
B_{22} &= (-1)^{2+2} \cdot \det \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} = -7 \\
B_{23} &= (-1)^{2+3} \cdot \det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = 3 \\
B_{31} &= (-1)^{3+1} \cdot \det \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = 1 \\
B_{32} &= (-1)^{3+2} \cdot \det \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = -5 \\
B_{33} &= (-1)^{3+3} \cdot \det \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = 3
\end{align*}
\]

Now we can write $B^{-1} = -\frac{1}{6} \begin{pmatrix} -3 & 3 & -3 \\ 5 & -7 & 3 \\ 1 & -5 & 3 \end{pmatrix}^T = \begin{pmatrix} 1/2 & -5/6 & -1/6 \\ -1/2 & 7/6 & 5/6 \\ 1/2 & -1/2 & -1/2 \end{pmatrix}$.

3.17) a) Show that if $A \in M^{3,3}$ is upper triangular, i.e. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ then $A^{-1}$ is upper triangular, i.e. $A^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}$. b) Find inverse of: $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}$.

Solution. a) Given a matrix that is upper (or lower) triangular, if $ij$, $i \neq j$ is a position where the given triangular matrix has a non-zero entry, then in the corresponding $ij$-th cofactor we get to evaluate the determinant of a matrix that has a zero row or a zero column. For example, for the given matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$, we have

\[
\begin{align*}
A_{12} &= (-1)^{1+2} \det \begin{pmatrix} 0 & a_{23} \\ 0 & a_{33} \end{pmatrix} = 0, \\
A_{13} &= (-1)^{1+3} \det \begin{pmatrix} 0 & a_{22} \\ 0 & 0 \end{pmatrix} = 0 \\
\text{and} \quad A_{23} &= (-1)^{2+3} \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} = 0.
\end{align*}
\]

Since the inverse matrix is a multiple of the adjoint matrix, and the latest is the transpose of the cofactor matrix, we get that the inverse matrix of an upper or lower triangular matrix is of the same form.

b) The given matrix $B$ is lower triangular. From the above observation, we have $B_{21} = B_{31} = B_{32} = 0$. We will evaluate the remaining cofactors:

\[
\begin{align*}
B_{11} &= (-1)^{1+1} \det \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = 1
\end{align*}
\]
\[ B_{12} = (-1)^{1+2} \text{det} \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} = -3 \]

\[ B_{13} = (-1)^{1+3} \text{det} \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} = 7 \]

\[ B_{22} = (-1)^{2+2} \text{det} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 1 \]

\[ B_{23} = (-1)^{2+3} \text{det} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = -3 \]

\[ B_{33} = (-1)^{3+3} \text{det} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = 1. \]

To evaluate the determinant of a triangular matrix like \( B \), it is enough to take the product of the diagonal elements, thus \( \text{det} B = 1 \). Now we use the formula \( B^{-1} = \frac{1}{\text{det} B} \text{adj} B \) and write

\[
B^{-1} = 1 \cdot \begin{pmatrix} 1 & -3 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -3 & 1 \end{pmatrix}.
\]

3.19) Given \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Find \( \text{adj} A \). When does \( A = \text{adj} A \)? Show that \( \text{adj}(\text{adj} A) = A \). Find \( A^{-1} \).

Solution. The evaluation of the cofactor matrix of a \( 2 \times 2 \) matrix is very simple. Given \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \):

\[
A_{11} = (-1)^{1+1}d = d, \quad A_{12} = (-1)^{1+2}c = -c, \quad A_{21} = (-1)^{2+1}b = -b, \quad A_{22} = (-1)^{2+2}a = a.
\]

Thus \( \text{adj} A = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \). We deduce that \( A = \text{adj} A \) if and only if \( a = d \) and \( b = c = 0 \), or, in other words, that \( A = \text{adj} A \) if and only if \( A \in \text{span} \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \} \).

We observe that \( \text{adj} A \) is a matrix that with respect to \( A \) has exchanged entries in the diagonal and opposite-sign entries in the remaining two positions. It is clear that repeating this pattern in finding \( \text{adj}(\text{adj} A) \) will lead to the original matrix \( A \).

To find the inverse of \( A \) we use the formula \( A^{-1} = \frac{1}{\text{det} A} \text{adj} A \), and since \( \text{det} A = ab - cd \), we get

\[
A^{-1} = \frac{1}{ab - cd} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\]

3.21) Given the matrix \( A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \) and the vector \( b = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \), find the inverse of the matrix \( A \) to solve the system \( Ax = b \).

Solution. First of all we verify that \( A \) is invertible, we know that a matrix is invertible if and only if is regular if and only if has a non-zero determinant. Since \( \text{det} A = 2 + 0 + 0 + 2 - 0 - 0 = 4 \), \( A \) is invertible. Thus we may solve the given equation multiplying both sides of the equality by \( A^{-1} \) on the left, so that we get \( A^{-1} \cdot Ax = A^{-1} \cdot b \), from which \( x = A^{-1} \cdot b \). We proceed evaluating the cofactor matrix:

\[
A_{11} = (-1)^{1+1} \text{det} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2
\]

\[
A_{12} = (-1)^{1+2} \text{det} \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} = 0
\]

\[
A_{13} = (-1)^{1+3} \text{det} \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} = 2
\]

\[
A_{21} = (-1)^{2+1} \text{det} \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} = 2
\]
\[ A_{22} = (-1)^{2+2} \det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2 \]
\[ A_{23} = (-1)^{2+3} \det \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} = 2 \]
\[ A_{31} = (-1)^{3+1} \det \begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix} = -2 \]
\[ A_{32} = (-1)^{3+2} \det \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0 \]
\[ A_{33} = (-1)^{3+3} \det \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} = 2. \]

We now evaluate the inverse of \( A \) using the formula \( A^{-1} = \frac{1}{\det A} \text{adj} A \) and we get
\[ A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 2 \\ -2 & 0 & 2 \end{pmatrix}^T = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \]

To conclude, we evaluate the matrix product and get the result of the given equation:
\[ x = A^{-1} \cdot b = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 + 0 - 1 \\ 0 + 0 + 0 \\ 3 + 0 + 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}. \]

3.23) Determine for what value of \( a \in \mathbb{R} \) the following matrix is regular and for those value find the inverse matrix:
\[ A = \begin{pmatrix} a + 1 & 1 & -a \\ 1 & a & 1 \\ 0 & -1 & a \end{pmatrix}. \]

**Solution.** A square matrix is regular if and only if its determinant is non-zero, thus we evaluate the determinant of the given matrix:
\[ \det A = \det \begin{pmatrix} a + 1 & 1 & -a \\ 1 & a & 1 \\ 0 & -1 & a \end{pmatrix} = a^2(a + 1) + a(a + 1) - a = (a + 1)(a^2 + 1). \]

Thus, if \( a \neq -1 \) the given matrix is regular. We will find the inverse of \( A \) using the formula \( A^{-1} = \frac{1}{\det A} \text{adj} A \). We start evaluating the cofactors of \( A \).
\[ A_{11} = (-1)^2 \det \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix} = a^2 + 1 \]
\[ A_{12} = (-1)^3 \det \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} = -a \]
\[ A_{13} = (-1)^4 \det \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix} = -1 \]
\[ A_{21} = (-1)^3 \det \begin{pmatrix} 1 & -a \\ -1 & a \end{pmatrix} = 0 \]
\[ A_{22} = (-1)^4 \det \begin{pmatrix} a + 1 & -a \\ 0 & a \end{pmatrix} = a(a + 1) \]
\[ A_{23} = (-1)^5 \det \begin{pmatrix} a + 1 & 1 \\ 0 & -1 \end{pmatrix} = a + 1 \]
\[ A_{31} = (-1)^4 \det \begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} = a^2 + 1 \]
\[ A_{32} = (-1)^5 \det \begin{pmatrix} a + 1 & -a \\ 1 & 1 \end{pmatrix} = -2a - 1 \]
Solution. The given matrix equation can be transformed as follows:

\[
A_{31} = (-1)^6 \det \begin{pmatrix} a + 1 & 1 \\ 1 & a \end{pmatrix} = a^2 + a - 1.
\]

Thus \(A^{-1} = \frac{1}{(a+1)(a^2+17)} \begin{pmatrix} a^2+1 & -a \\ 0 & a(a+1) \end{pmatrix} \cdot\begin{pmatrix} a^2+1 & -a \\ -2a-1 & a+1 \end{pmatrix} + \begin{pmatrix} a^2+1 & -a \\ -1 & a+1 \end{pmatrix} \).

3.25) Use an inverse matrix to solve the matrix equation \(XA = (X+I)B\) where \(I\) is the identity matrix,

\[
A = \begin{pmatrix} 6 & 9 & 1 \\ 4 & 2 & 4 \\ -1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 7 & 1 \\ 2 & 0 & 3 \\ -1 & 1 & 2 \end{pmatrix}.
\]

Solution. The given matrix equation can be transformed as follows:

\[
XA = (X+I)B \iff XA = XB + B \iff XA - XB = B \iff X(A - B) = B
\]

If the matrix \(A - B\) is invertible, then the latest equation can be solved by multiplying both sides of the equality by \((A - B)^{-1}\) on the right, so to get \(X = B(A - B)^{-1}\). Let’s evaluate \(A - B\) and its determinant, to establish if this approach can be used.

\[
A - B = \begin{pmatrix} 6 & 9 & 1 \\ 4 & 2 & 4 \\ -1 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 5 & 7 & 1 \\ 2 & 0 & 3 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & -1 & -3 \end{pmatrix}.
\]

Since \(\det \begin{pmatrix} 6 & 9 & 1 \\ 4 & 2 & 4 \\ -1 & 0 & -1 \end{pmatrix} = -6 + 1 + 12 = 7\), we proceed to find its inverse. Let’s evaluate the cofactors:

\[
(A - B)_{11} = (-1)^2 \det \begin{pmatrix} 2 & 1 \\ -1 & -3 \end{pmatrix} = -5
\]

\[
(A - B)_{12} = (-1)^3 \det \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} = 6
\]

\[
(A - B)_{13} = (-1)^4 \det \begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix} = -2
\]

\[
(A - B)_{21} = (-1)^3 \det \begin{pmatrix} 2 & 0 \\ -1 & -3 \end{pmatrix} = 6
\]

\[
(A - B)_{22} = (-1)^4 \det \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} = -3
\]

\[
(A - B)_{23} = (-1)^5 \det \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = 1
\]

\[
(A - B)_{31} = (-1)^4 \det \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = 2
\]

\[
(A - B)_{32} = (-1)^5 \det \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = -1
\]

\[
(A - B)_{33} = (-1)^6 \det \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = -2.
\]

Now we use the formula \((A - B)^{-1} = \frac{1}{\det(A - B)} \adj(A - B)\) and write

\[
(A - B)^{-1} = \frac{1}{7} \begin{pmatrix} -5 & 6 & -2 \\ 6 & -3 & 1 \\ 2 & -1 & -2 \end{pmatrix} \cdot\begin{pmatrix} -5 & 6 & 2 \\ 6 & -3 & 1 \\ -2 & 1 & -2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -5 & 6 & 2 \\ 6 & -3 & 1 \\ -2 & 1 & -2 \end{pmatrix}.
\]

To conclude, we evaluate the matrix product and get the result of the given matrix equation:

\[
X = B(A - B)^{-1} = \frac{1}{7} \begin{pmatrix} 5 & 7 & 1 \\ 2 & 0 & 3 \\ -1 & 1 & 2 \end{pmatrix} \cdot\begin{pmatrix} -5 & 6 & 2 \\ 6 & -3 & 1 \\ -2 & 1 & -2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -25 & 42 & -2 \\ 30 & -21 & 1 \\ 10 & -7 & 2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 15 & 10 & 1 \\ 7 & 15 & -2 \\ 7 & -7 & -5 \end{pmatrix}.
\]
4.1) Write down all solutions of the following homogeneous system

\[
\begin{align*}
    x - 2y + 3z &= 0 \\
    x + 2y - z &= 0.
\end{align*}
\]

4.2) Write down all solutions of the following homogeneous system

\[
\begin{align*}
    x - 2y + 4z &= 0 \\
    x + y - 2z &= 0.
\end{align*}
\]

4.3) Write down all solutions of the following homogeneous system for any possible value of \(a \in \mathbb{R}\)

\[
\begin{align*}
    x - 2y + z + 2w &= 0 \\
    -y + 2z + w &= 0 \\
    -2x + y + z - aw &= 0 \\
    ax - y + 2z + 7w &= 0.
\end{align*}
\]

4.4) Write down all solutions of the following homogeneous system for any possible value of \(a \in \mathbb{R}\)

\[
\begin{align*}
    x + 2y + z + 2w &= 0 \\
    2x - 3y + 2w &= 0 \\
    3x + 5y - z + w &= 0 \\
    4x + 2y + 6z + aw &= 0.
\end{align*}
\]

4.5) Use Gaussian elimination to find the general solution to the system:

\[
\begin{align*}
    x + y + w &= 1 \\
    2x + 2y + z + w &= 1 \\
    2x + 2y + 2w &= 2.
\end{align*}
\]

4.6) Use Gaussian elimination to find the general solution to the system:

\[
\begin{align*}
    x + 2y + 5z &= 20 \\
    3x - y + 2z &= 7 \\
    x - 5y - 8w &= -33.
\end{align*}
\]

4.7) Use Cramer’s rule to solve the system

\[
\begin{align*}
    x + 2y + z &= 3 \\
    2x + 5y - z &= -4 \\
    3x - 2y - z &= 5.
\end{align*}
\]

4.8) Use Cramer’s rule to solve the system

\[
\begin{align*}
    2x - 5y + 2z &= 2 \\
    x + 2y - 4z &= 5 \\
    3x - 4y - 6z &= 1.
\end{align*}
\]

4.9) Write down all solutions of the following system as the sum of a particular solution of the nonhomogeneous system with the linear space of solutions of the homogeneous system

\[
\begin{align*}
    x + 2y + 4z &= 5 \\
    2x - y + 3z &= 5 \\
    -x + 3y + z &= 0.
\end{align*}
\]

4.10) Write down all solutions of the following system as the sum of a particular solution of the nonhomogeneous system with the linear space of solutions of the homogeneous system

\[
\begin{align*}
    x + y + z &= 2 \\
    3x - y + z &= 0.
\end{align*}
\]
4.11) Write down all solutions of the following system as the sum of a particular solution of the nonhomogeneous system with the linear space of solutions of the homogeneous system

\[
\begin{align*}
    x - 2y + z - w &= 1 \\
    x + y - z + w &= 2 \\
    2x - y + z - w &= 1.
\end{align*}
\]

4.12) Write down all solutions of the following system as the sum of a particular solution of the nonhomogeneous system with the linear space of solutions of the homogeneous system

\[
\begin{align*}
    4x + y + z + 4w &= 33 \\
    5x - 4y + 2z - w &= 18 \\
    2x - 3y + z - 2w &= 1 \\
    x + 2y + 3w &= 16.
\end{align*}
\]

4.13) Determine for what value of the parameter \( a \) the following system has a unique solution, infinitely many solutions or no solution. Write down all solutions if any.

\[
\begin{align*}
    x + ay - 3z &= 5 \\
    ax - 3y + z &= 10 \\
    x + 9y - 10z &= a + 3.
\end{align*}
\]

4.14) Determine for what value of the parameter \( a \) the following system has a unique solution, infinitely many solutions or no solution. Write down all solutions if any.

\[
\begin{align*}
    x + y + az &= a \\
    x + ay + z &= 1 \\
    ax + y + z &= 1.
\end{align*}
\]

4.15) Determine for what value of the parameter \( a \) the following system has a unique solution, infinitely many solutions or no solution. Write down all solutions if any.

\[
\begin{align*}
    x + y - az &= 1 \\
    x - 2y + 3z &= 2 \\
    x + ay - z &= 1.
\end{align*}
\]

4.16) Determine for what value of the parameter \( a \) the following system has a unique solution, infinitely many solutions or no solution. Write down all solutions if any.

\[
\begin{align*}
    x + 2y + 2w &= 1 \\
    2x + 3y + 2z + w &= 2 \\
    x + 3y - 2z + aw &= 1.
\end{align*}
\]

4.17) Find the solutions of the following system, depending on the value of the parameter \( \lambda \in \mathbb{R} \):

\[
\begin{align*}
    x + y + \lambda z &= \lambda \\
    x + \lambda y + z &= 1 \\
    \lambda x + y + z &= 1.
\end{align*}
\]

4.18) Find the solutions to the following system, depending on the value of the parameter \( \lambda \in \mathbb{R} \):

\[
\begin{align*}
    \lambda x + y + z &= 1 \\
    x + \lambda y + z &= \lambda \\
    x + y + \lambda z &= \lambda^2.
\end{align*}
\]

4.19) Write down all solutions of the following system for any possible value of \( a, b \in \mathbb{R} \)

\[
\begin{align*}
    -2x + y + z &= 7 \\
    ax + 2y - z &= 2 \\
    -3x + y + 2z &= -b.
\end{align*}
\]

4.20) Write down all solutions of the following system for any possible value of \( a, b \in \mathbb{R} \)

\[
\begin{align*}
    x + y - z &= b \\
    -x + y + z &= 2 \\
    -2y + az &= 1.
\end{align*}
\]
4.1) Write down all solutions of the following homogeneous system

\[
\begin{align*}
x - 2y + 3z &= 0 \\
x + 2y - z &= 0.
\end{align*}
\]

Solution. The augmented matrix associated to the system is \[
\begin{pmatrix}
1 & -2 & 3 & 0 \\
1 & 2 & -1 & 0
\end{pmatrix}.
\]

We apply the matrix form of the Gaussian elimination, we start with the operation \(R_2 - R_1\) obtaining \[
\begin{pmatrix}
1 & -2 & 3 & 0 \\
0 & 4 & -4 & 0
\end{pmatrix}.
\]

We get the matrix to the echelon form with the operation \(R_2/4\) (dividing the last row by four):

\[
\begin{pmatrix}
1 & -2 & 3 & 0 \\
0 & 1 & -1 & 0
\end{pmatrix}.
\]

Since the rank of the matrix is two and the number of unknowns is three, we see that the dimension of the space of all solutions in one. It is enough to guess one solution of the system associated to the last augmented matrix, this system is equivalent to the given one

\[
\begin{align*}
x - 2y + 3z &= 0 \\
y - z &= 0.
\end{align*}
\]

Starting from the last equation, putting \(z = 1\), we get \(y = 1\), these values substituted into the first equation give \(x = -1\). Thus every solution \(x = (x, y, z)\) of the system can be expressed as a multiple of the one we have found, i.e. \(x \in \text{span}\{(-1, 1, 1)\}\).

Observation. This method is equivalent to the usual one, where we introduce parameters for the free variables, but it is often much more convenient. Anyway, in case we prefer, after we find the echelon form of the matrix and the associated system, we may conclude as follows: choosing \(z\) as a free variable, \(z = \alpha, \alpha \in \mathbb{R}\), from the second equation we get \(y = \alpha\), that substituted into the first equation give \(x = -\alpha\). Thus we see that every solution of the given system is of the form \((x, y, z) = (-\alpha, \alpha, \alpha) = \alpha(-1, 1, 1), \alpha \in \mathbb{R}\).

4.3) Write down all solutions of the following homogeneous system for any possible value of \(a \in \mathbb{R}\)

\[
\begin{align*}
x - 2y + z + 2w &= 0 \\
-y + 2z + w &= 0 \\
-2x + y + z - aw &= 0 \\
ax - y + 2z + 7w &= 0.
\end{align*}
\]

Solution. The augmented matrix of the given system is:

\[
\begin{pmatrix}
1 & -2 & 1 & 2 & 0 \\
0 & -1 & 2 & 1 & 0 \\
-2 & 1 & 1 & -a & 0 \\
  & -1 & 2 & 7 & 0
\end{pmatrix}.
\]

Considering the presence of the parameter, we prefer to avoid a Gaussian elimination. In this case it is possible because the given system has four equations and four unknowns, thus the matrix associated to the system is a square one.

\[
A = \begin{pmatrix}
1 & -2 & 1 & 2 \\
0 & -1 & 2 & 1 \\
-2 & 1 & 1 & -a \\
  & -1 & 2 & 7
\end{pmatrix}.
\]

An \(n \times n\) system of linear equations has a unique solution if and only if the determinant of the associated matrix is non-zero. Evaluating the determinant of \(A\), we will be able to select the particular values of \(a\) for which the given system does not have a unique solution. We evaluate the determinant along the first column:

\[
\det A = (-1)^{1+1}(1) \det \begin{pmatrix}
-1 & 2 & 1 \\
1 & 1 & -a \\
-1 & 2 & 7
\end{pmatrix} + (-1)^{3+1}(-2) \det \begin{pmatrix}
-2 & 1 & 2 \\
-1 & 2 & 1 \\
-1 & 2 & 7
\end{pmatrix} + (-1)^{4+1}(a) \det \begin{pmatrix}
-2 & 1 & 2 \\
-1 & 2 & 1 \\
  & 1 & -a
\end{pmatrix} =
\]

\[
= -18 - 2(-18) - a(3a - 3) = -3a^2 + 3a + 18 = -3(a^2 - a - 6).
\]

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Thus, the determinant of \( \mathbf{A} \) is equal to zero only for \( a = -2 \), or \( a = 3 \). For all possible values of \( a \in \mathbb{R} \), \( a \neq -2 \) and \( a \neq 3 \), the given homogeneous system has a unique solution, the trivial one \( (x, y, z, w) = (0, 0, 0, 0) \). Let’s now analyse the case when \( a = -2 \). For this value of the parameter the augmented matrix is

\[
\begin{pmatrix}
1 & -2 & 1 & 2 & 0 \\
0 & -1 & 2 & 1 & 0 \\
0 & -3 & 3 & 6 & 0 \\
0 & -2 & 1 & 5 & 0
\end{pmatrix}.
\]

We proceed with the matrix form of Gaussian elimination, starting with the row operations \( R_3 + 2R_1 \), \( R_4 - 4R_3 \), we get

\[
\begin{pmatrix}
1 & -2 & 1 & 2 & 0 \\
0 & -1 & 2 & 1 & 0 \\
0 & -3 & 3 & 6 & 0 \\
0 & -2 & 1 & 5 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & -2 & 1 & 2 & 0 \\
0 & 1 & -2 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 3 & -3 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & -2 & 1 & 2 & 0 \\
0 & 1 & -2 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Since the rank of the matrix is three and the number of unknowns is four, we see that the dimension of the space of all solutions in one. It is enough to guess one solution of the system associated to the last augmented matrix, this system is equivalent to the given one

\[
x - 2y + z + 2w = 0 \\
y - 2z - w = 0 \\
z - w = 0.
\]

Starting from the last equation, putting \( w = 1 \), we get \( z = 1 \), these values substituted into the second equation give \( y = 3 \), and finally from the first equation we get \( x = 3 \). Thus every solution \( \mathbf{x} = (x, y, z, w) \) of the system can be expressed as a multiple of the one we have found, i.e. \( \mathbf{x} \in \text{span}\{(3, 3, 1, 1)\} \).

Let’s now analyse the case when \( a = 3 \). For this value of the parameter the augmented matrix is

\[
\begin{pmatrix}
1 & -2 & 1 & 2 & 0 \\
0 & -1 & 2 & 1 & 0 \\
-2 & 1 & 1 & -3 & 0 \\
3 & -1 & 2 & 7 & 0
\end{pmatrix}.
\]

We proceed with the matrix form of Gaussian elimination, starting with the row operations \( R_3 + 2R_1 \), \( R_4 - 4R_3 \), we get

\[
\begin{pmatrix}
1 & -2 & 1 & 2 & 0 \\
0 & -1 & 2 & 1 & 0 \\
0 & -3 & 3 & 1 & 0 \\
0 & 5 & -1 & 1 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & -2 & 1 & 2 & 0 \\
0 & 1 & -2 & -1 & 0 \\
0 & 0 & 3 & 2 & 0 \\
0 & 0 & 9 & 6 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & -2 & 1 & 2 & 0 \\
0 & 1 & -2 & -1 & 0 \\
0 & 0 & 3 & 2 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Since the rank of the matrix is three and the number of unknowns is four, we see that the dimension of the space of all solutions in one. It is enough to guess one solution of the system associated to the last augmented matrix, this system is equivalent to the given one

\[
x - 2y + z + 2w = 0 \\
y - 2z - w = 0 \\
3z + 2w = 0.
\]

Starting from the last equation, putting \( w = 3 \), we get \( z = -2 \), these values substituted into the second equation give \( y = -1 \), and finally from the first equation we get \( x = -6 \). Thus every solution \( \mathbf{x} = (x, y, z, w) \) of the system can be expressed as a multiple of the one we have found, i.e. \( \mathbf{x} \in \text{span}\{(6, 1, 2, -3)\} \).

4.5) Use Gaussian elimination to find the general solution to the system:

\[
x + y + w = 1 \\
2x + 2y + z + w = 1 \\
2x + 2y + 2w = 2.
\]

**Solution.** The augmented matrix associated to the system is

\[
\begin{pmatrix}
1 & 1 & 0 & 1 & 1 \\
2 & 2 & 1 & 1 & 1 \\
2 & 2 & 0 & 2 & 2
\end{pmatrix}.
\]
We apply the matrix form of the Gaussian elimination, we start with the operations $R_2 - 2R_1$, $R_3 - R_2$ obtaining

\[
\begin{pmatrix}
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 \\
0 & 0 & -1 & 1 & 1 \\
\end{pmatrix}
+ R_3 \sim
\begin{pmatrix}
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Since the rank of the augmented matrix is equal to the rank of the matrix associated to the homogeneous system, the system has solutions. Since the rank is two and the number of unknowns is four, we see that the dimension of the space of all solutions of the homogeneous system associated to the given one is two. We consider the system associated to the last augmented matrix, this is equivalent to the given one,

\[
x + y + w = 1 \\
z - w = -1.
\]

Choosing as free variables $w$ and $y$, and setting $w = \alpha$ and $y = \beta$, with $\alpha, \beta \in \mathbb{R}$, we get from the last equation $z = -1 + \alpha$, and from the first $x = 1 - \alpha - \beta$. Thus any solution of the given system can be written as:

\[
x = (x, y, z, w) = (1 - \alpha - \beta, \beta, -1 + \alpha, \alpha) = (1, 0, -1, 0) + \alpha(-1, 0, 1, 1) + \beta(-1, 1, 0, 0), \quad \alpha, \beta \in \mathbb{R},
\]

or equivalently

\[
x = (1, 0, -1, 0) + \text{span}\{(-1, 0, 1, 1), (-1, 1, 0, 0)\},
\]

where $\text{span}\{(-1, 0, 1, 1), (-1, 1, 0, 0)\}$ is the space of all solutions of the homogeneous system associated to the given one.

4.7) Use Cramer’s rule to solve the system

\[
x + 2y + z = 3 \\
2x + 5y - z = -4 \\
3x - 2y - z = 5.
\]

**Solution.** The matrix associated to the homogeneous system is

\[
A = \begin{pmatrix}
1 & 2 & 1 \\
2 & 5 & -1 \\
3 & -2 & -1 \\
\end{pmatrix}.
\]

In order to use Cramer’s rule, we need to verify that $\det A \neq 0$. Indeed $\det A = -5 - 6 - 15 - 2 + 4 = -28$. Thus, Cramer’s rule tells us that the system has a unique solution that can be computed as follows:

\[
x = \frac{\det C_1}{\det A} = \frac{\det \begin{pmatrix}
3 & 2 & 1 \\
-4 & 5 & -1 \\
-2 & -1 & -1 \\
\end{pmatrix}}{-28} = \frac{-15 - 10 + 8 - 25 - 6 - 8}{-28} = \frac{-56}{-28} = 2
\]

\[
y = \frac{\det C_2}{\det A} = \frac{\det \begin{pmatrix}
1 & 3 & 1 \\
2 & -4 & -1 \\
3 & 5 & -1 \\
\end{pmatrix}}{-28} = \frac{4 - 9 + 10 + 12 + 5 + 6}{-28} = \frac{28}{-28} = -1
\]

\[
z = \frac{\det C_3}{\det A} = \frac{\det \begin{pmatrix}
1 & 2 & 3 \\
2 & 5 & -4 \\
3 & -2 & 5 \\
\end{pmatrix}}{-28} = \frac{25 - 24 - 12 - 45 - 8 - 20}{-28} = \frac{-84}{-28} = 3.
\]

The unique solution of the system is $(x, y, z) = (2, -1, 3)$.

4.9) Write down all solutions of the following system as the sum of a particular solution of the nonhomogeneous system with the linear space of solutions of the homogeneous system

\[
x + 2y + 4z = 5 \\
2x - y + 3z = 5 \\
-x + 3y + z = 0.
\]
Solution. The matrix associated to the homogeneous system is

\[
A = \begin{pmatrix}
1 & 2 & 4 \\
2 & -1 & 3 \\
-1 & 3 & 1
\end{pmatrix}.
\]

Evaluating \( \det A = -1 - 6 + 24 - 4 - 9 - 4 = 0 \), we verify that we cannot use Cramer’s rule to solve the system. The augmented matrix of the system is

\[
\begin{pmatrix}
1 & 2 & 4 & 5 \\
2 & -1 & 3 & 5 \\
-1 & 3 & 1 & 0
\end{pmatrix}.
\]

Let’s start a matrix form of Gaussian reduction: with operations \( R_2 - 2R_1 \), and \( R_3 + R_1 \) we get:

\[
\begin{pmatrix}
1 & 2 & 4 & 5 \\
0 & -5 & -5 & -5 \\
0 & 5 & 5 & 5
\end{pmatrix} \sim
\begin{pmatrix}
1 & 2 & 4 & 5 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Since the rank of the augmented matrix is equal to the rank of the matrix associated to the homogeneous system, the system has solutions. Considering that the rank is two and the number of unknowns is three, we see that the dimension of the space of all solutions of the homogeneous system is one. We consider the system associated to the last augmented matrix, this is equivalent to the given one,

\[
\begin{align*}
x + 2y + 4z &= 5 \\
y + z &= 1.
\end{align*}
\]

We now may proceed in two ways, a guessing method, or introducing a parameter.

Guessing method. We know that the structure of the solution set of our system is: “a particular solution plus the space of solutions of the homogeneous system”. Knowing that the dimension of the space of solutions of the homogeneous system is one, we may guess one non-trivial solution \( x_h \) of the homogeneous system and a particular solution \( x_p \) of the system. Looking for \( x_h \), we set \( z = 1 \) in the last (homogeneous) equation and get \( y = -1 \). These values, substituted into the first (homogeneous) equation, give \( x = -2 \). Thus \( x_h = (-2, -1, 1) \).

Looking for \( x_p \), we set \( y = 0 \) in the last (this time nonhomogeneous) equation and get \( z = 1 \), these values, substituted into the first equation, give \( x = 1 \). Thus \( x_p = (1, 0, 1) \). All solutions of the given system can be written as

\[ x = x_p + \text{span}\{x_h\} = (1, 0, 1) + \text{span}\{(-2, -1, 1)\}. \]

Introducing a parameter. We choose \( z \) as free variable, \( z = \alpha \), \( \alpha \in \mathbb{R} \). From the last equation we thus get \( y = 1 - \alpha \). Substituting into the first equation, we get \( x = 3 - 2\alpha \). Thus any solution of the system can be written as

\[ x = (3 - 2\alpha, 1 - \alpha, \alpha) = (3, 1, 0) + \alpha(-2, -1, 1) = (3, 1, 0) + \text{span}\{(-2, -1, 1)\}. \]

Again we recovered the structure of the solution set of the system, as \( y = (3, 1, 0) \) is a particular solution of the system, and \( \text{span}\{(-2, -1, 1)\} \) is the space of solutions of the homogeneous system.

4.11) Write down all solutions of the following system as the sum of a particular solution of the nonhomogeneous system with the linear space of solutions of the homogeneous system

\[
\begin{align*}
x - 2y + z - w &= 1 \\
x + y - z + w &= 2 \\
2x - y + z - w &= 1.
\end{align*}
\]

Solution. The matrix associated to the homogeneous system is

\[
A = \begin{pmatrix}
1 & -2 & 1 & -1 \\
1 & 1 & -1 & 1 \\
2 & -1 & 1 & -1
\end{pmatrix}
\]

and the augmented matrix is

\[
B = \begin{pmatrix}
1 & -2 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 2 \\
2 & -1 & 1 & -1 & 1
\end{pmatrix}.
\]
Since $A$ is not a square matrix, we proceed with a Gaussian reduction of $B$. We start with operations $R_2 - R_1$, $R_3 - 2R_1$, and we get
\[
\begin{pmatrix}
1 & -2 & 1 & -1 \\
0 & 3 & -2 & 2 \\
0 & 3 & -1 & 1
\end{pmatrix} \sim
\begin{pmatrix}
1 & -2 & 1 & -1 \\
0 & 3 & -2 & 2 \\
0 & 0 & 1 & -1 & -2
\end{pmatrix}.
\]

The rank of the augmented matrix is equal to the rank of the matrix associated to the homogeneous system, thus the system has solutions. Since the rank is three and the number of unknowns is four, we see that the dimension of the space of all solutions of the homogeneous system (associated to the given one) is one. We know that the structure of the solution set of our system is: $x_p + \text{span}\{x_h\}$, where $x_p$ is a particular solution of the system, $x_h$ is a solution of the homogeneous system, and $\text{span}\{x_h\}$ is the space of solutions of the homogeneous system. We consider the system associated to the last augmented matrix, this is equivalent to the homogeneous system.

$x = 2y + z - w = 1$
$3y - 2z + 2w = 1$
$z - w = -2.$

We will now guess $x_h$, a non trivial solution of the homogeneous system. Starting from the last (homogeneous) equation, we set $w = 1$ and get $z = 1$. These values, substituted into the second (homogeneous) equation, give $y = 0$ and finally from the first (homogeneous) equation we get $x = 0$. Thus $x_h = (0, 0, 1, 1)$. We now guess $x_p$: starting again from the last equation, we set $w = 1$ and get $z = -1$. These values, substituted into the second equation, give $y = -1$ and finally from the first equation we get $x = 1$. Thus $x_p = (1, -1, -1, 1)$. Any solutions of the given system can be written as
\[x = x_p + \text{span}\{x_h\} = (1, -1, -1, 1) + \text{span}\{(0, 0, 1, 1)\}.
\]

Introducing a parameter. We choose $w$ as a free variable, $w = \alpha$, $\alpha \in \mathbb{R}$. From the last equation we get $z = -2 + \alpha$. Substituting into the second and then into the first equation, we get $y = -1$ and then $x = 1$. Thus, for any possible $\alpha \in \mathbb{R}$, we have solution of the system
\[x = (1, -1, -2 + \alpha, \alpha) = (1, -1, -2, 0) + \alpha(0, 0, 1, 1) = (1, -1, -2, 0) + \text{span}\{(0, 0, 1, 1)\}.
\]

It is important to recognise that we get the same solutions as in the previous case, because $(1, -1, -2, 0)$ is only a different particular solution that can be obtain from the $x_p$ found before, by subtracting once the found solution $x_h$ of the homogeneous system.

4.13) Determine for what value of the parameter $a$ the following system has a unique solution, infinitely many solutions or no solution. Write down all solutions if any.

\[
\begin{align*}
x + ay - 3z &= 5 \\
a x - 3y + z &= 10 \\
x + 9y - 10z &= a + 3.
\end{align*}
\]

**Solution.** The matrix associated to the homogeneous system is
\[
A = \begin{pmatrix}
1 & a & -3 \\
a & -3 & 1 \\
1 & 9 & -10
\end{pmatrix}.
\]

Since $A$ is a square matrix, we will evaluate $\det A$ to distinguish the values of the parameter $a$ for which we can use Cramer’s rule ($\det A \neq 0$).

\[
\det A = 30 + a - 27 - 9 - 9 + 10a^2 = 2(5a^2 - 13a + 6).
\]

Consequently, $\det A = 0$ if and only if $a = 2$ or $a = \frac{3}{5}$. For any possible $a \in \mathbb{R}$, $a \neq 2$, $a \neq \frac{3}{5}$, the given system has a unique solution that can be evaluated using Cramer’s rule:
\[
x = \frac{\det C_1}{\det A} = \frac{\det \begin{pmatrix}
5 & a & -3 \\
a + 3 & -3 & 1 \\
a + 3 & 9 & -10
\end{pmatrix}}{2(5a^2 - 13a + 6)} = \frac{150 + a(a + 3) - 270 - 9(a + 3) - 45 + 100a}{2(5a^2 - 13a + 6)} = \frac{a^2 + 94a - 192}{2(5a^2 - 13a + 6)}
\]

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Solution.

Determine for what value of the parameter $a$ the following system has a unique solution, infinitely many solutions or no solution. Write down all solutions if any.

We introduce a parameter, choosing $z$ as a free variable, $z = \alpha$, $\alpha \in \mathbb{R}$, from the last equation we get $y = \alpha$. These values, substituted into the first equation, give $x = 5 + \alpha$. Thus, any solution of the system can be written as

$$x = (x, y, z) = (5 + \alpha, \alpha, \alpha) = (5, 0, 0) + \alpha(1, 1, 1), \quad \alpha \in \mathbb{R}$$

or equivalently

$$\mathbf{x} = (5, 0, 0) + \text{span}\{(5, 1, 1)\},$$

where $(5, 0, 0)$ is a particular solution of the system, and \text{span}\{(5,1,1)\} is the space of solutions of the homogeneous system.

For $a = \frac{3}{5}$, we will perform a Gaussian reduction starting from the augmented matrix of the given system, where we substitute the value $a = \frac{3}{5}$:

$$\begin{pmatrix}
\frac{3}{5} & -3 & 5 \\
-3 & 1 & 10 \\
1 & -10 & \frac{12}{5}
\end{pmatrix} R_2 - \frac{3}{5} R_1 \sim
\begin{pmatrix}
1 & 2 & -3 & 5 \\
0 & -\frac{42}{5} & -7 & -\frac{7}{5} \\
0 & -\frac{42}{5} & -7 & -\frac{7}{5}
\end{pmatrix} R_3 - R_1 \sim
\begin{pmatrix}
1 & 2 & -3 & 5 \\
0 & 84 & -14 & -7 \\
0 & 0 & 0 & \frac{164}{5}
\end{pmatrix} R_3 + 5 R_2 \sim
\begin{pmatrix}
1 & 2 & -3 & 5 \\
0 & 0 & 0 & \frac{164}{5}
\end{pmatrix}.$$

The rank of the augmented matrix is three, while the rank of the matrix associated to the homogeneous system is two, thus, by Frobenius’ theorem, the system has no solutions.

**4.15)** Determine for what value of the parameter $a$ the following system has a unique solution, infinitely many solutions or no solution. Write down all solutions if any.

$$\begin{align*}
x + y - az &= 1 \\
x - 2y + 3z &= 2 \\
x + ay - z &= 1.
\end{align*}$$

Solution. The matrix associated to the homogeneous system is

$$\mathbf{A} = \begin{pmatrix}
1 & 1 & -a \\
1 & -2 & 3 \\
1 & a & -1
\end{pmatrix}.$$

Since $\mathbf{A}$ is a square matrix, we will evaluate $\det \mathbf{A}$ to distinguish the values of the parameter $a$ for which we can use Cramer’s rule ($\det \mathbf{A} \neq 0$).

$$\det \mathbf{A} = 2 + 3 - a^2 - 2a - 3a + 1 = -a^2 - 5a + 6 = (a + 6)(1 - a)$$

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Consequently, \( \det \mathbf{A} = 0 \) if and only if \( a = -6 \) or \( a = 1 \). For any possible \( a \in \mathbb{R}, a \neq -6, a \neq 1 \), the given system has a unique solution that can be evaluated using Cramer’s rule:

\[
\begin{align*}
x &= \frac{\det \mathbf{C}_1}{\det \mathbf{A}} = \frac{\begin{vmatrix} 1 & 1 & -a \\ 2 & -2 & 3 \\ 1 & a & -1 \end{vmatrix}}{(a + 6)(1 - a)} = \frac{2 + 3 - 2a^2 - 2a - 3a + 2}{(a + 6)(1 - a)} = \frac{(1 - a)(2a + 7)}{(a + 6)(1 - a)} = \frac{2a + 7}{a + 6} \\
y &= \frac{\det \mathbf{C}_2}{\det \mathbf{A}} = \frac{\begin{vmatrix} 1 & 1 & -a \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{vmatrix}}{(a + 6)(1 - a)} = \frac{-2 + 3 - a + 2a - 3 + 1}{(a + 6)(1 - a)} = \frac{a - 1}{(a + 6)(1 - a)} = \frac{-1}{a + 6} \\
z &= \frac{\det \mathbf{C}_3}{\det \mathbf{A}} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & a & 1 \end{vmatrix}}{(a + 6)(1 - a)} = \frac{-2 + 2 + a + 2 - 2a - 1}{(a + 6)(1 - a)} = \frac{-a + 1}{(a + 6)(1 - a)} = \frac{1}{a + 6}.
\end{align*}
\]

For \( a = -6 \), we will perform a Gaussian reduction starting from the augmented matrix of the given system, where we substitute the value \( a = -6 \):

\[
\begin{pmatrix}
1 & 1 & 6 & 1 \\
1 & -2 & 3 & 2 \\
1 & -6 & -1 & 1
\end{pmatrix}
\sim
\begin{pmatrix}
R_2 - R_1 & 1 & 6 & 1 \\
0 & -3 & -3 & 1 \\
0 & -7 & -7 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 1 & 6 & 1 \\
0 & -3 & -3 & 1 \\
3R_3 - R_2 & 0 & 0 & 0 & -7
\end{pmatrix}.
\]

The rank of the augmented matrix is three, while the rank of the matrix associated to the homogeneous system is two, thus, by Frobenius’ theorem, the system has no solutions.

For \( a = 1 \), we will perform a Gaussian reduction starting from the augmented matrix of the given system, where we substitute the value \( a = 1 \):

\[
\begin{pmatrix}
1 & 1 & -1 & 1 \\
1 & -2 & 3 & 2 \\
1 & 1 & -1 & 1
\end{pmatrix}
\sim
\begin{pmatrix}
R_2 - R_1 & 1 & 6 & 1 \\
0 & -3 & -3 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

The rank of the augmented matrix is equal to the rank of the matrix associated to the homogeneous system, thus the system has solutions. Since the rank is two and the number of unknowns is three, we see that the dimension of the space of all solutions of the homogeneous system is one. We consider the system associated to the last augmented matrix, this is equivalent to the given one,

\[
\begin{align*}
x + y - z &= 1 \\
-3y + 2z &= 1.
\end{align*}
\]

We introduce a parameter, choosing \( z \) as a free variable, \( z = 3\alpha, \alpha \in \mathbb{R} \), from the last equation we get

\[
y = 2\alpha - \frac{1}{7}.
\]

These values, substituted into the first equation, give \( x = \alpha + \frac{4}{7} \). Thus, any solution of the system can be written as

\[
\mathbf{x} = (x, y, z) = (\alpha + \frac{4}{3}, 2\alpha - \frac{1}{3}, 3\alpha) = (\frac{4}{3}, -\frac{1}{3}, 0) + \alpha(1, 2, 3), \quad \alpha \in \mathbb{R}
\]

or equivalently

\[
\mathbf{x} = (\frac{4}{3}, -\frac{1}{3}, 0) + \text{span}\{(1, 2, 3)\},
\]

where \((\frac{4}{3}, -\frac{1}{3}, 0)\) is a particular solution of the system, and \(\text{span}\{(1, 2, 3)\}\) is the space of solutions of the homogeneous system.

**4.17** Find the solutions of the following system, depending on the value of the parameter \( \lambda \in \mathbb{R} \):

\[
\begin{align*}
x + y + \lambda z &= \lambda \\
x + \lambda y + z &= 1 \\
\lambda x + y + z &= 1.
\end{align*}
\]

**Solution.** The matrix associated to the homogeneous system is

\[
\mathbf{A} = \begin{pmatrix}
1 & 1 & \lambda \\
1 & \lambda & 1 \\
\lambda & 1 & 1
\end{pmatrix}.
\]
Since $A$ is a square matrix, we will evaluate $\det A$ to distinguish the values of the parameter $\lambda$ for which we can use Cramer’s rule ($\det A \neq 0$).

$$\det A = \lambda + \lambda - \lambda^3 - 1 - 1 = - (\lambda^3 - 3\lambda + 2).$$

By looking at the matrix $A$, we notice that, for $\lambda = 1$, the matrix has three identical rows, thus $\lambda = 1$ must be a value for which $\det A = 0$. In order to find other possible values of $\lambda$ that make the determinant equal to zero, we perform the following division of polynomials:

$$\frac{(\lambda^3 - 3\lambda + 2)}{(\lambda - 1)} = \lambda^2 + \lambda - 2$$

$$-\frac{\lambda^2 - 3\lambda + 2}{\lambda - 1} = - (\lambda^2 - \lambda)$$

$$-\frac{-2\lambda + 2}{-2\lambda + 2} = 0$$

Consequently, $\det A = -(\lambda - 1)(\lambda^2 + \lambda - 2) = -(\lambda - 1)^2(\lambda + 2)$. Thus $\det A = 0$ if and only if $\lambda = 1$ or $\lambda = -2$. For any possible $\lambda \in \mathbb{R}$, $\lambda \neq 1$, $\lambda \neq -2$, the given system has a unique solution that can be evaluated using Cramer’s rule:

$$x = \frac{\det C_1}{\det A} = \frac{\det \begin{pmatrix} \lambda & 1 & \lambda \\ 1 & \lambda & 1 \\ 1 & 1 & 1 \end{pmatrix}}{-(\lambda - 1)^2(\lambda + 2)} = \frac{0}{-(\lambda - 1)^2(\lambda + 2)} = 0$$

$$y = \frac{\det C_2}{\det A} = \frac{\det \begin{pmatrix} 1 & \lambda & \lambda \\ 1 & 1 & 1 \\ \lambda & 1 & 1 \end{pmatrix}}{-(\lambda - 1)^2(\lambda + 2)} = \frac{0}{-(\lambda - 1)^2(\lambda + 2)} = 0$$

$$z = \frac{\det C_3}{\det A} = \frac{\det \begin{pmatrix} 1 & 1 & \lambda \\ 1 & 1 & \lambda \\ \lambda & 1 & 1 \end{pmatrix}}{-(\lambda - 1)^2(\lambda + 2)} = \frac{\det A}{-(\lambda - 1)^2(\lambda + 2)} = 1.$$

For $\lambda = 1$, we will perform a Gaussian reduction starting from the augmented matrix of the given system, where we substitute the value $\lambda = 1$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} R_2 - R_1 \\ R_3 - R_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank of the augmented matrix is equal to the rank of the matrix associated to the homogeneous system, thus the system has solutions. Since the rank is one and the number of unknowns is three, we see that the dimension of the space of all solutions of the homogeneous system is two. We consider the "system" associated to the last augmented matrix, this is equivalent to the given one, and it is composed by a single equation:

$$x + y + z = 1.$$

This time we choose the guessing method. We know that the structure of the solution set of our system is: $x_p + \text{span}\{x_{h1}, x_{h2}\}$, where $x_p$ is a particular solution of the last equation, $x_{h1}$ and $x_{h2}$ are two linearly independent solutions of the homogeneous equation, and thus $\text{span}\{x_h\}$ is the space of all solutions of the homogeneous equation. We easily guess $x_{h1} = (1, -1, 0)$, $x_{h2} = (0, 1, -1)$, and $x_p = (1, 0, 0)$. Thus, all solutions of the system can be written as

$$x = (x, y, z) = (1, 0, 0) + \text{span}\{(1, -1, 0), (0, 1, -1)\}.$$

For $\lambda = -2$, we will perform a Gaussian reduction starting from the augmented matrix of the given system, where we substitute the value $\lambda = -2$:

$$\begin{pmatrix} 1 & 1 & -2 & -2 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} R_2 - R_1 \\ R_3 + 2R_1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 & -2 \\ 0 & -3 & 3 & 5 \\ 0 & 3 & -3 & -3 \end{pmatrix} \sim \begin{pmatrix} R_3 + R_2 \\ R_2 + 2R_1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 & -2 \\ 0 & -3 & 3 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$
The rank of the augmented matrix is three, while the rank of the matrix associated to the homogeneous system is two, thus, by Frobenius’ theorem, the system has no solutions.

4.19) Write down all solutions of the following system for any possible value of \(a, b \in \mathbb{R}\)

\[
\begin{align*}
-2x + y + z &= 7 \\
ax + 2y - z &= 2 \\
-3x + y + 2z &= -b.
\end{align*}
\]

**Solution.** The matrix associated to the homogeneous system is

\[
A = \begin{pmatrix}
-2 & 1 & 1 \\
a & 2 & -1 \\
-3 & 1 & 2
\end{pmatrix}.
\]

Since \(A\) is a square matrix, we will evaluate \(\det A\) to distinguish the values of the parameter \(a\) for which we can use Cramer’s rule (\(\det A \neq 0\)).

\[
\det A = -8 + 3 + a + 6 - 2a = -a - 1.
\]

Consequently, \(\det A = 0\) if and only if \(a = -1\). For any possible \(a \in \mathbb{R}\), \(a \neq -1\), the given system has a unique solution that can be evaluated using Cramer’s rule:

\[
x = \frac{\det C_1}{\det A} = \frac{28 + b + 2 + 2b + 7 - 4}{-a - 1} = \frac{3b + 33}{-a - 1},
\]

\[
y = \frac{\det C_2}{\det A} = \frac{-8 + 21 - ab + 6 + 2b - 14a}{-a - 1} = \frac{2b - ab - 14a + 19}{-a - 1},
\]

\[
z = \frac{\det C_3}{\det A} = \frac{4b - 6 + 7a + 42 + 4 + ab}{-a - 1} = \frac{4b + ab + 7a + 40}{-a - 1}.
\]

For \(a = -1\), we will perform a Gaussian reduction starting from the augmented matrix of the given system, where we substitute the value \(a = -1\):

\[
\begin{pmatrix}
-2 & 1 & 1 & 7 \\
-1 & 2 & -1 & 2 \\
-3 & 1 & 2 & -b
\end{pmatrix} \to \begin{pmatrix}
1 & -2 & 1 & -2 \\
0 & -3 & 3 & 3 \\
0 & -5 & 5 & -b - 6
\end{pmatrix} \sim \begin{pmatrix}
1 & -2 & 1 & -2 \\
0 & 1 & -1 & -1 \\
0 & 0 & 0 & 3b + 33
\end{pmatrix}.
\]

We now see that the rank of the augmented matrix depends on the value of \(b\). If \(b \neq -11\), \(3b + 33 \neq 0\), then the rank of the augmented matrix is three, while the rank of the matrix associated to the homogeneous system is two. By Frobenius’s theorem, the system has no solutions. If \(b = -11\), then the system has infinitely many solutions. The rank of the augmented matrix (equal to the rank of the matrix associated to the homogeneous system) is two and the number of unknowns is three, thus the dimension of the space of all solutions of the homogeneous system is one. We consider the system associated to the last augmented matrix, this is equivalent to the given one,

\[
-x - 2y + z = 2 \\
y - z = 1.
\]

We introduce a parameter, choosing \(z\) as a free variable, \(z = \alpha, \alpha \in \mathbb{R}\), from the last equation we get \(y = \alpha - 1\). These values, substituted into the first equation, give \(x = \alpha - 4\). Thus, any solution of the system can be written as

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
\alpha - 4 \\
\alpha - 1 \\
\alpha
\end{pmatrix} = (-4, -1, 0) + \alpha(1, 1, 1), \quad \alpha \in \mathbb{R}
\]

or equivalently

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = (-4, -1, 0) + \text{span}\{(1, 1, 1)\},
\]

where \((-4, -1, 0)\) is a particular solution of the system, and \(\text{span}\{(1, 1, 1)\}\) is the space of solutions of the homogeneous system.

Let’s summarise the final answer to the given exercise. If \(a \neq -1\), for any possible \(b\), the system has a unique solution \(\begin{pmatrix}
\frac{3b+33}{a-1} \\
\frac{2b-ab-14a+19}{a-1} \\
\frac{4b+ab+7a+40}{a-1}
\end{pmatrix}\). If \(a = -1\), and \(b \neq -11\), the system has no solution. If \(a = -1\), and \(b = -11\), the system has infinitely many solutions \(\begin{pmatrix}
-4 \\
-1 \\
0
\end{pmatrix} + \alpha(1, 1, 1), \quad \alpha \in \mathbb{R}\).
5.1) Determine if the following are linear transformations:
   a) \( f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = x + y, \)
   b) \( f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = x + 1, \)
   c) \( f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = xy, \)
   d) \( f : \mathbb{R}^2 \to \mathbb{R}, \ f(ax^2 + bx + c) = (a + b + c), \)
   e) \( f : \mathbb{R}^2 \to \mathbb{P}^1, \ f(a, b) = b + ax. \)

5.2) Determine if the following transformation \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) is linear:
   a) \( f(x, y, z) = (2x - y, y - 1, x + 2y), \)
   b) \( f(x, y, z) = (2x - y, y - x, xy), \)
   c) \( f(x, y, z) = (2x - y^2 + z, y + 10z, x + y), \)
   d) \( f(x, y, z) = (2x - y, y - x - \sqrt{3}z, x + 2y). \)

5.3) Given the linear transformation \( l : \mathbb{R}^2 \to \mathbb{R}, \ l(x, y) = (2x - 2y, -x + y), \) write the matrix associated to \( l \) with respect to the standard basis of \( \mathbb{R}^2, \) find \( \text{Ker}(l), \text{Im}(l), \) their bases and dimensions. Find all vectors of \( \mathbb{R}^2 \) that are mapped to \((4, -2).\) Determine if \( l \) is invertible and, if possible, find its inverse.

5.4) Given the transformation \( l : \mathbb{R}^3 \to \mathbb{R}^3, \ l(x, y, z) = (y + z, x - z, x + y + z), \) write the matrix associated to \( l \) with respect to the standard basis of \( \mathbb{R}^3, \) find \( \text{Ker}(l), \text{Im}(l), \) their bases and dimensions. Find all vectors of \( \mathbb{R}^3 \) that are mapped to \((4, 2, -1).\) Determine if \( l \) is invertible and, if possible, find its inverse.

5.5) Given \( l : \mathbb{R}^3 \to \mathbb{R}^3, \ l(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, 4x_1 + 5x_2 + 6x_3, x_1 + x_2 + x_3), \) find \( \text{Ker}(l), \text{Im}(l), \) their bases and dimensions.

5.6) Given \( l : \mathbb{R}^3 \to \mathbb{R}^2, \ l(x_1, x_2, x_3) = (2x_1 - x_2 + 3x_3, x_1 + x_2 + x_3), \) find \( \text{Ker}(l), \text{Im}(l), \) their bases and dimensions.

5.7) Determine the linear transformation \( l : \mathbb{R}^3 \to \mathbb{R}^2 \) defined by:
   \[
   l(1, 1, 0) = (2, -1), \quad l(0, 1, 2) = (1, 1), \quad l(2, 0, 0) = (-1, -3).
   \]
   Find \( \text{Ker}(l), \) its basis and dimension. Calculate \( l(1, 2, -2). \)

5.8) Determine the linear transformation \( l : \mathbb{R}^2 \to \mathbb{R}^3 \) defined by:
   \[
   l(2, 3) = (-2, 1, 2), \quad l(1, -1) = (0, 3, 2).
   \]
   Find \( \text{Ker}(l), \text{Im}(l) \) their basis and dimension. Calculate \( l(4, 11). \)

5.9) Given the linear transformation \( l : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by \( l(1, 2, 3) = (-3, -8, -3), \ l(1, 1, 0) = (1, 5, 2) \) and such that \( \text{Ker}(l) = \text{span}\{\!(1, 1, 1)\!\}, \) find the matrix associated with \( l \) with respect to the standard basis, find \( \text{Im}(l), \) its basis and dimension. Find all \( \mathbf{v} \in \mathbb{R}^3 \) such that \( l(\mathbf{v}) = (2, 3, 1).\)

5.10) Given \( l : \mathbb{R}^4 \to \mathbb{R}^3 \) such that \( l(1, 1, -1, 0) = (0, 0, 0), \ l(1, 2, -1, 2) = (-1, -3, 1), \ l(1, 0, 0, -1) = (0, 0, 0), \ l(1, 1, 1, 1) = (5, 8, 2), \) find the matrix \( \mathbf{A} \) associated with \( l \) with respect to the standard bases. Find all \( \mathbf{v} \in \mathbb{R}^4 \) such that \( \mathbf{A}\mathbf{v} = (13, 21, 5).\)

5.11) Given the linear transformation \( l : \mathbb{R}^3 \to \mathbb{P}^1, \ l(a, b, c) = b - c + (2a - c)x, \) find the matrix associated with \( l \) with respect to the standard bases, evaluate \( l(2, 1, -1), \) find \( \text{Ker}(l) \) its basis and dimension. Is \( l \) surjective?

5.12) Given \( l : \mathcal{M}^{2,2} \to \mathcal{P}^3 \) defined by \( l(\mathbf{A}) = l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + (2a - b)x + (b + c)x^2 + (a - b + c + d)x^3, \) find the matrix associated with \( l \) with respect to the standard bases. Is \( l \) an isomorphism?

5.13) Given the matrix \( \mathbf{M} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \) we define the linear transformation \( l : \mathcal{M}^{2,2} \to \mathcal{M}^{2,2} \) by \( l(\mathbf{X}) = \mathbf{M} \cdot \mathbf{X}. \) Is \( l \) an isomorphism? If possible, find its inverse.
5.14) Given the linear transformation \( l : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) that has as associated matrix with respect to the standard bases \( A(l) = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{pmatrix} \), write down the general form of \( l(x, y, z, w) \), find \( \text{Ker}(l) \), \( \text{Im}(l) \), their bases and dimensions.

5.15) Given the linear transformations
\[
f : \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad f(x, y, z) = (2x - y, 2y + z), \quad \text{and} \quad g : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad g(u, v) = (u, u + v, u - v),
\]
find the matrix associated to \( f \circ g \) and \( g \circ f \) with respect to the standard basis. Find \( \text{rank}(f \circ g) \) and \( \text{rank}(g \circ f) \), is one of the two compositions an isomorphism? If yes find its inverse, otherwise identify Kernel and Image.

5.16) Given the linear transformations
\[
f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(x, y) = (4x - y, x - 2y), \quad \text{and} \quad g : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad g(1, 1) = (-1, -2), \quad g(0, 1) = (2, 4)
\]
find the matrix associated to \( f \circ g \) and \( g \circ f \) with respect to the standard basis. Find \( \text{rank}(f \circ g) \) and \( \text{rank}(g \circ f) \), is one of the two compositions an isomorphism? If yes find its inverse, otherwise identify Kernel and Image.

5.17) For the space \( \mathbb{R}^3 \) are given two ordered bases:
\[
B = \{(1, 0, 0), (1, 2, 0), (1, 2, 1)\}, \quad \text{and} \quad S = \{(2, -2, 2), (2, -1, -2), (0, 2, -2)\}.
\]
Find \( P_{B \rightarrow S} \) the matrix of transition from basis \( B \) to basis \( S \).
(Hint: use \( C \), the standard basis of \( \mathbb{R}^3 \), and the fact that \( P_{B \rightarrow S} = P_{S \rightarrow C}^{-1}P_{B \rightarrow C} \).

5.18) In \( \mathbb{R}^3 \) are given the standard basis \( C = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \) and the bases
\[
B = \{(1, 1, 0), (0, 1, 0), (0, -1, 1)\}, \quad \text{and} \quad D = \{(1, 0, 1), (1, 1, 1), (0, -1, 1)\}.
\]
Find the transition matrices \( P_{B \rightarrow C}, \quad P_{D \rightarrow C}, \quad P_{B \rightarrow D}, \quad P_{D \rightarrow B} \).

5.19) Let \( A(l) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 2 \end{pmatrix} \) be the matrix associated to a linear transformation \( l : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) with respect to the standard bases of \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \). Find the matrix associated to the given transformation with respect to the bases \( S \) and \( B \), where:
\[
S = \{(2, -2, 2), (2, -1, -2), (0, 2, -2)\}, \quad \text{and} \quad B = \{(1, 1), (1, -1)\}.
\]

5.20) Given \( l : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that \( l(x, y) = (4x - 2y, 2x + y) \), find the matrix associated to \( l \) with respect to the basis \( F = \{(1, 1), (-1, 0)\} \).

5.21) Given the linear transformation \( l_h : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad l_h(x, y, z) = (2x + y, y - z, 2y + hz) \), write the matrix associated to \( l_h \) with respect to the standard basis of \( \mathbb{R}^3 \), find \( \text{Ker}(l_h) \), \( \text{Im}(l_h) \), their bases and dimensions for every possible value of \( h \in \mathbb{R} \). Does there exist a value of \( h \) for which \( l_h \) is an isomorphism?.

5.22) Given the transformation \( l_h : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by \( l_h(x, y, z) = (x - hz, x + y - hz, -hx + z) \), where \( h \in \mathbb{R} \) is a parameter. Find, for all possible values of \( h \), \( \text{Ker}(l_h) \), \( \text{Im}(l_h) \), their bases and dimensions.
Determine \( l_h^{-1}(1, 0, 1) = \{(x, y, z) \in \mathbb{R}^3 : l_h(x, y, z) = (1, 0, 1)\} \).
5.1) Determine if the following are linear transformations:

a) \( f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = x + y, \)

b) \( f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = x + 1, \)

c) \( f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = xy, \)

d) \( f : \mathbb{P}^2 \to \mathbb{P}^2, \ f(ax^2 + bx + c) = (a + b, b + c), \)

e) \( f : \mathbb{R}^2 \to \mathbb{P}^1, \ f(a, b) = b + a^2x. \)

Solution. Given two linear spaces \( L \) and \( M \) a function \( f : L \to M \) is called linear if the following two conditions are satisfied:

i) \( f(\mathbf{v} + \mathbf{u}) = f(\mathbf{v}) + f(\mathbf{u}), \) for every \( \mathbf{v}, \mathbf{u} \in L, \)

ii) \( f(\alpha \mathbf{v}) = \alpha f(\mathbf{v}), \) for every \( \mathbf{v} \in L, \ \alpha \in \mathbb{R}. \)

a) Given \( f : \mathbb{R}^2 \to \mathbb{R}, \) defined as \( f(x, y) = x + y, \) we prove that \( f \) is linear. Indeed, given any \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^2, \ \mathbf{u} = (x_1, y_1), \ \mathbf{v} = (x_2, y_2), \) and \( \alpha \in \mathbb{R}: \)

i) \( f(\mathbf{v} + \mathbf{u}) = f((x_1, y_1) + (x_2, y_2)) = f(x_1 + x_2, y_1 + y_2) = x_1 + x_2 + y_1 + y_2 = (x_1 + y_1) + (x_2 + y_2) = f(x_1, y_1) + f(x_2, y_2) = f(\mathbf{v}) + f(\mathbf{u}), \)

ii) \( f(\alpha \mathbf{v}) = f(\alpha (x_1, y_1)) = \alpha x_1 + \alpha y_1 = \alpha x_1 + (x_1 + y_1) = \alpha f(x_1, y_1) = \alpha f(\mathbf{v}). \)

b) Given \( f : \mathbb{R}^2 \to \mathbb{R}, \) defined as \( f(x, y) = x + 1, \) we prove that \( f \) is not linear. Indeed, given \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^2, \ \mathbf{u} = (x_1, y_1), \ \mathbf{v} = (x_2, y_2) \), in general condition i) is not satisfied, i.e. we have

i) \( f(\mathbf{v} + \mathbf{u}) = f((x_1, y_1) + (x_2, y_2)) = f(x_1 + x_2, y_1 + y_2) = x_1 + x_2 + 1 \neq (x_1 + 1) + (x_2 + 1) = f(x_1, y_1) + f(x_2, y_2) = f(\mathbf{v}) + f(\mathbf{u}), \)

it is enough to verify this inequality for \( \mathbf{v} = \mathbf{u} = (1, 2), \) when we get

\[ f(\mathbf{v} + \mathbf{u}) = f(2, 4) = 2 + 1 = 3, \] while \( f(\mathbf{v}) + f(\mathbf{u}) = f(1, 2) + f(1, 2) = 1 + 1 + 1 + 1 = 4. \)

c) Given \( f : \mathbb{R}^2 \to \mathbb{R}, \) defined as \( f(x, y) = xy, \) we prove that \( f \) is not linear. Indeed, it is easy to disprove condition i): we need to find \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^2, \ \mathbf{u} = (x_1, y_1), \ \mathbf{v} = (x_2, y_2), \) and prove that

i) \( f(\mathbf{v} + \mathbf{u}) = f((x_1, y_1) + (x_2, y_2)) = f(x_1 + x_2, y_1 + y_2) = (x_1 + x_2)(y_1 + y_2) \neq (x_1 y_1 + x_2 y_2) = f(x_1, y_1) + f(x_2, y_2) = f(\mathbf{v}) + f(\mathbf{u}), \)

it is enough to choose \( \mathbf{v} = \mathbf{u} = (1, 2), \) then we get

\[ f(\mathbf{v} + \mathbf{u}) = f(2, 4) = 2 \cdot 4 = 8, \] while \( f(\mathbf{v}) + f(\mathbf{u}) = f(1, 2) + f(1, 2) = 1 \cdot 2 + 1 \cdot 2 = 4. \)

d) Given \( f : \mathbb{P}^2 \to \mathbb{R}^2, \) defined as \( f(ax^2 + bx + c) = (a + b, b + c), \) we prove that \( f \) is linear. Indeed, given any \( \mathbf{u}, \mathbf{v} \in \mathbb{P}^2, \ \mathbf{u} = a_1 x^2 + b_1 x + c_1, \ \mathbf{v} = a_2 x^2 + b_2 x + c_2, \) and \( \alpha \in \mathbb{R}: \)

i) \( f(\mathbf{v} + \mathbf{u}) = f((a_1 x^2 + b_1 x + c_1) + (a_2 x^2 + b_2 x + c_2)) = f((a_1 + a_2) x^2 + (b_1 + b_2) x + (c_1 + c_2)) = (a_1 + a_2 + b_1 + b_2, b_1 + b_2 + c_1 + c_2) = ((a_1 + b_1) + (a_2 + b_2), (b_1 + c_1) + (b_2 + c_2)) = (a_1 + b_1 + c_1) + (a_2 + b_2, b_2 + c_2) = f(a_1 x^2 + b_1 x + c_1) + f(a_2 x^2 + b_2 x + c_2) = f(\mathbf{v}) + f(\mathbf{u}), \)

ii) \( f(\alpha \mathbf{v}) = f(\alpha a_1 x^2 + \alpha b_1 x + \alpha c_1) = (\alpha a_1 + \alpha b_1, \alpha b_1 + \alpha c_1) = \alpha (a_1 + b_1, b_1 + c_1) = \alpha f(\mathbf{v}). \)

e) Given \( f : \mathbb{R}^2 \to \mathbb{P}^1, \) defined as \( f(a, b) = b + a^2x, \) we prove that \( f \) is not linear. Indeed, we will disprove condition ii): we need to find \( \mathbf{v} = (a, b) \in \mathbb{R}^2 \) and \( \alpha \in \mathbb{R} \) such that

ii) \( f(\alpha \mathbf{v}) = f(\alpha a, \alpha b) = \alpha b + (\alpha a)^2 x \neq \alpha (b + a^2 x) = \alpha f(\mathbf{v}) = f(\mathbf{v}). \)

It is enough to choose \( \mathbf{v} = (1, 2), \) and \( \alpha = 3, \) then we get

\[ f(\alpha \mathbf{v}) = f(3, 6) = 6 + 9x^2, \] while \( f(\mathbf{v}) = 3f(1, 2) = 3(2 + x^2) = 6 + 3x^2. \)
5.3) Given the linear transformation \( l : \mathbb{R}^2 \to \mathbb{R}^2, l(x,y) = (2x - 2y, -x + y) \), write the matrix associated to \( l \) with respect to the standard basis of \( \mathbb{R}^2 \), find \( \ker l, \text{Im} l \), their bases and dimensions. Find all vectors of \( \mathbb{R}^2 \) that are mapped to \((4, -2)\). Determine if \( l \) is invertible and, if possible, find its inverse.

**Solution.** Given two linear spaces \( L, M \) with finite dimension, and two respective bases \( B_L = \{b_1, b_2, \ldots, b_n\}, B_M = \{e_1, e_2, \ldots, e_m\} \), every linear transformation \( l : L \to M \) is uniquely determined by a matrix \( A = A(l, B_L, B_M) = (a_{ij}) \in \mathcal{M}^{m,n} \), where \( a_{ij} \) are as follows:

\[
l(b_1) = a_{11}c_1 + a_{21}c_2 + \cdots + a_{m1}c_m
\]
\[
l(b_2) = a_{12}c_1 + a_{22}c_2 + \cdots + a_{m2}c_m
\]
\[
\vdots
\]
\[
l(b_n) = a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{mn}c_m.
\]

Such matrix \( A \) has the property that, for every \( \mathbf{v} \in L \),

\[
[l(\mathbf{v})]_{B_M} = A \cdot [\mathbf{v}]_{B_L}
\]

where \([\mathbf{v}]_{B} \) are the coordinates of the vector \( \mathbf{v} \) with respect to the basis \( B \) seen as a column.

Following this definition, we are looking for the matrix \( A \) that represents the given linear transformation with respect to the canonical basis \( B = \{(1, 0), (0, 1)\} \) of \( \mathbb{R}^2 \) (both as starting and landing space: \( L = M = \mathbb{R}^2 \)).

We have:

\[
l(1,0) = (2,-1) \quad \text{and} \quad l(0,1) = (-2,1).
\]

Thus \( A = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \).

In order to find \( \ker l, \text{Im} l \), we recall this two fundamental definitions. Given a linear map \( l : L \to M \):

\[
\ker l = \{ \mathbf{v} \in L : l(\mathbf{v}) = \mathbf{0}_M \}, \quad (\ker l \text{ is a subspace of } L)
\]
\[
\text{Im} l = \{ \mathbf{u} \in M : \text{there exists } \mathbf{v} \in L, l(\mathbf{v}) = \mathbf{u} \}, \quad (\text{Im} l \text{ is a subspace of } M).
\]

In our case, \( \ker l = \left\{(x,y) \in \mathbb{R}^2 : A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \) is the space of solutions of the homogeneous system associated to \( A \). We use the matrix form of Gaussian reduction and get:

\[
\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_1/2 \quad R_1/2 + R_2} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.
\]

Thus, \( \ker l = \text{span}\{(1,1)\} \), dim \( \ker l = 1 \), and a basis of \( \ker l \) is \( B_K = \{(1,1)\} \).

Before we evaluate \( \text{Im} l \), we recall two fundamental Theorems. Given two finite dimensional linear spaces \( L, M \), any basis \( B_L = \{b_1, b_2, \ldots, b_n\} \) of \( L \), and a linear transformation \( l : L \to M \):

i) \( \text{Im} l = \text{span}\{l(b_1), l(b_2), \ldots, l(b_n)\} \),

ii) dim \( L = \text{dim} \ker l + \text{dim} \text{Im} l \).

In our case, ii) implies that dim \( \text{Im} l \) = dim \( \mathbb{R}^2 \) − dim \( \ker l = 1 \), and, from i) we get

\[
\text{Im} l = \text{span}\{(1,0), l(0,1)\} = \text{span}\{(2, -1), (-2,1)\} = \text{span}\{(2, -1)\}
\]

thus, a basis of \( \text{Im} l \) is \( B_I = \{(2, -1)\} \).

We now need to find all vectors of \( \mathbb{R}^2 \) that are mapped to \((4, -2)\), this means to identify the set \( \left\{(x,y) \in \mathbb{R}^2 : A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \right\} \), i.e. the set of solutions of the nonhomogeneous system with augmented matrix:

\[
\begin{pmatrix} 2 & -2 & 4 \\ -1 & 1 & -2 \end{pmatrix} \xrightarrow{R_1/2 \quad R_1/2 + R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.
\]

After the simple reduction, we see that the system has solutions, and, knowing already the space of solutions of the homogeneous system (\( \ker l \)), we just need to guess a particular solution \( x_p \) of the nonhomogeneous system, for example \( x_p = (3,1) \), to write

\[
\left\{(x,y) \in \mathbb{R}^2 : A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \right\} = x_p + \ker l = (3,1) + \text{span}\{(1,1)\}.
\]
A linear transformation $l : L \to M$ is invertible if and only if it is injective, if and only if $\text{Ker} \, l = \{0_L\}$. This implies that in our case, the given map $l$ is not invertible.

5.5) Given $l : \mathbb{R}^3 \to \mathbb{R}^3$, $l(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, 4x_1 + 5x_2 + 6x_3, x_1 + x_2 + x_3)$, find $\text{Ker}(l)$, $\text{Im}(l)$, their bases and dimensions.

**Solution.** We refer to the Solution of Exercise 3 for all relevant definitions and theorems.

We recall that a linear transformation is uniquely determined by the image of the vectors in a basis of the starting linear space. Thus the given conditions define a linear transformation $l$ associated to the given linear transformation with respect to $C$. For this purpose, we need to evaluate:

- $l(1,0,0) = (1,4,1)$
- $l(0,1,0) = (2,5,1)$
- $l(0,0,1) = (3,6,1)$.

Now we can write $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{pmatrix}$. To find $\text{Ker} \, l = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$, the space of solutions of the homogeneous system associated to $A$, we first evaluate $\det A$, because, in case $\det A \neq 0$, we know that the homogeneous system has a unique solution, the trivial one. Since $\det A = 5+12+12-15-6-8 = 0$, we will use the matrix form of Gaussian reduction and get:

$$
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
R_2 - 4R_1 \\
\sim \\
R_3 - R_1
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 0 \\
0 & -3 & -6 & 0 \\
0 & -1 & -2 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 2 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Setting $x_3 = \alpha, \alpha \in \mathbb{R}$, we get from the second row (imagining the correspondent equation $x_2 + 2x_3 = 0$) $x_2 = -2\alpha$, and, subsequently, from the first row $x_1 = \alpha$. Thus we conclude that:

$\text{Ker} \, l = \{(\alpha, -2\alpha, \alpha) \in \mathbb{R}^3 \} = \text{span}\{(1, -2, 1)\}, \text{dim Ker} \, l = 1$, and a basis of $\text{Ker} \, l$ is $B_K = \{(1, -2, 1)\}$.

From the Theorem on dimensions ii), we get $\dim \text{Im} \, l = \dim \mathbb{R}^3 - \dim \text{Ker} \, l = 2$, and, from Theorem i) we get

$$
\text{Im} \, l = \text{span}\{l(1,0,0), l(0,1,0), l(0,0,1)\} = \text{span}\{(1, 4, 1), (2, 5, 1), (3, 6, 1)\} = \text{span}\{(1, 4, 1), (2, 5, 1)\},
$$

where the first two vectors were chosen randomly, just considering that, because they are linearly independent (they are not a multiple of each other), they must span the space $\text{Im} \, l$ that has dimension 2. Thus, a basis of $\text{Im} \, l$ is $B_I = \{(1, 4, 1), (2, 5, 1)\}$.

5.7) Determine the linear transformation $l : \mathbb{R}^3 \to \mathbb{R}^2$ defined by:

- $l(1,1,0) = (2, -1)$, $l(0,1,2) = (1, 1)$, $l(2,0,0) = (-1, -3)$.

Find $\text{Ker}(l)$, its basis and dimension. Calculate $l(1, 2, -2)$.

**Solution.** We recall that a linear transformation is uniquely determined by the image of the vectors in a basis of the starting linear space. Thus the given conditions define a linear transformation $l : \mathbb{R}^3 \to \mathbb{R}^2$ if $B = \{(1, 1, 0), (0, 1, 2), (2, 0, 0)\}$ is a basis of $\mathbb{R}^3$. We can easily verify that the given vectors are linearly independent, by evaluating the determinant of the matrix $B$ formed by the vectors of the set $B$ written as columns:

$$
\det \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} = 4 \neq 0;
$$

thus, knowing that the dimension of $\mathbb{R}^3$ is three, we can conclude that the given three vectors form a basis of $\mathbb{R}^3$.

In order to find the matrix $A$ that represents the linear transformation $l$ with respect to the canonical basis $C$ of $\mathbb{R}^3$, $C = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, we start observing that such matrix $A$ has the property that:

- $\begin{pmatrix} 1 \\ 1 \end{pmatrix} A = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$,
- $\begin{pmatrix} 0 \\ 1 \end{pmatrix} A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,
\[
\begin{bmatrix}
2 \\
0 \\
0
\end{bmatrix}
= \begin{pmatrix}
-1 \\
-3
\end{pmatrix},
\]

that can be written, using the properties of matrix operations, in this compact way:

\[
A \begin{bmatrix}
1 & 0 & 2 \\
1 & 1 & 0 \\
0 & 2 & 0
\end{bmatrix} = AB = \begin{pmatrix}
2 & -1 & 2 \\
-1 & 1 & -3
\end{pmatrix}.
\]

From the last matrix equation, knowing that \(B\) is certainly invertible because its determinant is non-zero, it follows that

\[
A = \begin{pmatrix}
2 & -1 & 2 \\
-1 & 1 & -3
\end{pmatrix} B^{-1}.
\]

We now evaluate the inverse of \(B\) using the formula \(B^{-1} = \frac{1}{\det B} \text{adj} B\). We have already calculated the determinant of \(B\): \(\det B = 4\). We now proceed with the evaluation of \(\text{adj} B\), calculating all cofactors of \(B\):

\[
\begin{align*}
B_{11} &= (-1)^{1+1} \cdot \det \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = 0 \\
B_{12} &= (-1)^{1+2} \cdot \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \\
B_{13} &= (-1)^{1+3} \cdot \det \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} = 2 \\
B_{21} &= (-1)^{2+1} \cdot \det \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 4 \\
B_{22} &= (-1)^{2+2} \cdot \det \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 0 \\
B_{23} &= (-1)^{2+3} \cdot \det \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = -2 \\
B_{31} &= (-1)^{3+1} \cdot \det \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = -2 \\
B_{32} &= (-1)^{3+2} \cdot \det \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = 2 \\
B_{33} &= (-1)^{3+3} \cdot \det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1
\end{align*}
\]

Now we can write \(B^{-1} = \frac{1}{4} \begin{pmatrix} 0 & 0 & 2 \\ 4 & 0 & -2 \\ -2 & 2 & 1 \end{pmatrix}^T = \frac{1}{4} \begin{pmatrix} 0 & 4 & -2 \\ 0 & 0 & 2 \\ 2 & -2 & 1 \end{pmatrix}\). We thus evaluate \(A\), calculating the indicated product:

\[
A = \begin{pmatrix}
2 & -1 & 2 \\
-1 & 1 & -3
\end{pmatrix} B^{-1} = \frac{1}{4} \begin{pmatrix}
2 & -1 & 2 \\
-1 & 1 & -3
\end{pmatrix} \begin{pmatrix}
0 & 4 & -2 \\
0 & 0 & 2 \\
2 & -2 & 1
\end{pmatrix} = \frac{1}{4} \begin{pmatrix}
4 & 4 & -4 \\
-6 & 2 & 1
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\
-\frac{3}{2} & 1 & -1
\end{pmatrix}.
\]

The found matrix \(A\) represents the linear transformation \(l\) with respect to the canonical basis \(C\) of \(\mathbb{R}^3\), thus we may evaluate:

\[
l(x, y, z) = A \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \frac{1}{4} \begin{pmatrix}
4 & 4 & -4 \\
-6 & 2 & 1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = (x + y - z, -\frac{3}{2}x + \frac{1}{2}y + \frac{1}{4}z),
\]

from which:

\[
l(1, 2, -2) = (1, 1).
\]
To find \( \text{Ker} \ l = \{(x, y, z) \in \mathbb{R}^3 : A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \} \), the space of solutions of the homogeneous system associated to \( A \), we will use the matrix form of Gaussian reduction and get:

\[
\begin{pmatrix}
  1 & 1 & -1 \\
  -\frac{3}{2} & \frac{1}{2} & -\frac{1}{4} \\
  -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\
\end{pmatrix}
\begin{pmatrix}
  R_2 + \frac{2}{3} R_1 \\
  \sim \\
  0 & 2 & -\frac{5}{4} \\
\end{pmatrix}
\begin{pmatrix}
  1 \\
  0 \\
  -\frac{5}{4} \\
\end{pmatrix}.
\]

The rank of \( A \) is two, this implies that \( \dim \text{Ker} \ l = \dim \mathbb{R}^3 - \text{rank} \ A = 1 \). We may easily guess a solution of the homogeneous system associated to the last matrix, setting \( z = 8 \), from the second row we get \( y = 5 \), and substituting these values into the first row, we get \( x = 3 \). We conclude that

\[
\text{Ker} \ l = \text{span}\{(3, 5, 8)\}, \quad \text{and a basis of Ker} \ l \text{ is } \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}.
\]

5.9) Given the linear transformation \( l : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by \( l(1, 2, 3) = (-3, -8, -3) \), \( l(1, 1, 0) = (1, 5, 2) \) and such that \( \text{Ker} \ l = \text{span}\{(1, 1, 1)\} \), find the matrix associated with \( l \) with respect to the standard basis, find \( \text{Im} \ l \), its basis and dimension. Find all \( v \in \mathbb{R}^3 \) such that \( l(v) = (2, 3, 1) \).

**Solution.**

The linear transformation \( l \) is well defined if \( B = \{(1, 2, 3), (1, 1, 0), (1, 1, 1)\} \) is a basis of \( \mathbb{R}^3 \). We can easily verify if the three vectors are linearly independent by evaluating the determinant of the matrix \( B \) formed by the vectors of the set \( B \) written as columns:

\[
\det \begin{pmatrix}
  1 & 1 & 1 \\
  2 & 1 & 1 \\
  3 & 0 & 1 \\
\end{pmatrix} = 1 + 3 + 0 - 3 - 0 - 2 = -1 \neq 0;
\]

thus, knowing that the dimension of \( \mathbb{R}^3 \) is three, we can conclude that \( B \) is a basis of \( \mathbb{R}^3 \). In order to find the matrix \( A \) that represents the linear transformation \( l \) with respect to the canonical basis \( C \) of \( \mathbb{R}^3 \), \( C = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \), we start observing that such matrix \( A \) has the property that:

\[
A \begin{pmatrix}
  1 \\
  2 \\
  3 \\
\end{pmatrix} = \begin{pmatrix}
  -3 \\
  -8 \\
  -3 \\
\end{pmatrix},
\]

\[
A \begin{pmatrix}
  1 \\
  1 \\
  0 \\
\end{pmatrix} = \begin{pmatrix}
  1 \\
  5 \\
  2 \\
\end{pmatrix},
\]

\[
A \begin{pmatrix}
  1 \\
  1 \\
  1 \\
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0 \\
  0 \\
\end{pmatrix},
\]

that, using the properties of matrix operations, can be written in this compact way:

\[
A \begin{pmatrix}
  1 & 1 & 1 \\
  2 & 1 & 1 \\
  3 & 0 & 1 \\
\end{pmatrix} = AB = \begin{pmatrix}
  -3 & 1 & 0 \\
  -8 & 5 & 0 \\
  -3 & 2 & 0 \\
\end{pmatrix}.
\]

From the last matrix equation, knowing that \( B \) is certainly invertible because its determinant is non-zero, it follows that

\[
A = ABB^{-1} = \begin{pmatrix}
  -3 & 1 & 0 \\
  -8 & 5 & 0 \\
  -3 & 2 & 0 \\
\end{pmatrix} B^{-1}.
\]

We now evaluate the inverse of \( B \) using the formula \( B^{-1} = \frac{1}{\det B} \text{adj} B \). We have already calculated the determinant of \( B \): \( \det B = -1 \). We now proceed with the evaluation of \( \text{adj} B \), calculating all cofactors of \( B \):

\[
B_{11} = (-1)^{1+1} \cdot \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1
\]

\[
B_{12} = (-1)^{1+2} \cdot \det \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = 1
\]

\[
B_{13} = (-1)^{1+3} \cdot \det \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = -3
\]
Given the linear transformation \( l \) with respect to the canonical basis \( C \) of \( \mathbb{R}^3 \), thus we may evaluate:

Now we can write \( B^{-1} = - \begin{pmatrix} 1 & 1 & -3 \\ -1 & -2 & 3 \\ 0 & 1 & -1 \end{pmatrix}^T \) = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 2 & -1 \\ 3 & -3 & 1 \end{pmatrix} \). We thus evaluate \( A \), calculating the indicated product:

\[
A = \begin{pmatrix} -3 & 1 & 0 \\ -8 & 5 & 0 \\ -3 & 2 & 0 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} -3 & 1 & 0 \\ -8 & 5 & 0 \\ -3 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 0 \\ -1 & 2 & -1 \\ 3 & -3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & -1 \\ 3 & 2 & -5 \\ 1 & 1 & -2 \end{pmatrix}.
\]

The found matrix \( A \) represents the linear transformation \( l \) with respect to the canonical basis \( C \) of \( \mathbb{R}^3 \), thus we may evaluate:

\[
l(x, y, z) = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 3 & 2 & -5 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (2x - y - z, 3x + 2y - 5z, x + y - 2z).
\]

As previously explained in the Solution of Exercise 3, the column of \( A \), seen as vectors in \( \mathbb{R}^3 \), are the image of the vectors in the canonical basis \( B \) of \( \mathbb{R}^3 \). Thus, recalling Theorem i) from the solution of Exercise 3, we can write

\[
\text{Im} \, l = \text{span}\{(2, 3, 1), (-1, 2, 1), (-1, -5, -2)\}.
\]

Now, without using a Gaussian reduction to find a set of linearly independent vectors that have the same span, we may reason as follows. It was given at the beginning of the exercise the information that the dimension of \( \text{Ker} \, l \) is one, thus (Theorem ii) in Solution of Exercise 3) \( \dim \text{Im} \, l = \dim \mathbb{R}^3 - \dim \text{Ker} \, l = 2 \), we may choose any two linearly independent vectors among the vectors that form the column of \( A \) and these will span the \( \text{Im} \, l \). For example, \( \text{Im} \, l = \text{span}\{(2, 3, 1), (-1, 2, 1)\} \).

We now need to find all vectors \( v \) of \( \mathbb{R}^3 \) that are mapped to \( (2, 3, 1) \), this means to identify the set

\[
\{(x, y, z) \in \mathbb{R}^3 : A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\}, \quad \text{i.e. the set of solutions of the non homogeneous system with augmented matrix:}
\]

\[
\begin{pmatrix} 2 & -1 & 1 & 2 \\ 3 & 2 & -5 & 3 \\ 1 & 1 & -2 & 1 \end{pmatrix} \rightarrow R_1 \rightarrow R_3 \sim \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

We now see that the system has solutions, and, knowing already the space of solutions of the homogeneous system (\( \text{Ker} \, l \)), we just need to guess a particular solution \( x_p \) of the nonhomogeneous system, for example \( x_p = (1, 0, 0) \), to write:

\[
\{v \in \mathbb{R}^3 : l(v) = (2, 3, 1)\} = \{(x, y, z) \in \mathbb{R}^3 : A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\} = x_p + \text{Ker} \, l = (1, 0, 0) + \text{span}\{(1, 1, 1)\}.
\]

5.11) Given the linear transformation \( l : \mathbb{R}^3 \rightarrow \mathbb{P}^1 \), \( l(a, b, c) = b - c + (2a - c)x \), find the matrix associated with \( l \) with respect to the standard bases, evaluate \( l(2, 1, -1) \), find \( \text{Ker} \, l \) its basis and dimension. Is \( l \) surjective?
Solution. We will consider $C$, the standard basis of $\mathbb{R}^3$, $C = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, and $T$, the standard basis of $P^1$, $T = \{1, x\}$. Using the approach described at the beginning of Solution of Exercise 3, we create the matrix $A$ associated to $l : \mathbb{R}^3 \to P^1$, $l(a, b, c) = b - c + (2a - c)x$, with respect to $C$ and $T$ as follows:

$$
l(1, 0, 0) = 2x = 0(1) + 2(x),
$$
$$
l(0, 1, 0) = 1 = 1(1) + 0(x),
$$
$$
l(0, 0, 1) = -1 - x = -1(1) - 1(x),
$$
from which

$$
A = A(l, C, T) = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix}.
$$

To evaluate $l(2, 1, -1)$ we just use the definition of $l$ with $(a, b, c) = (2, 1, -1)$: $l(2, 1, -1) = 2 + 5x$.

To find $\ker l = \{(x, y, z) \in \mathbb{R}^3 : A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$, the space of solutions of the homogeneous system associated to $A$, we will use the matrix form of Gaussian reduction and get:

$$
\begin{pmatrix} 0 & 1 & -1 & 0 \\ 2 & 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.
$$

The rank of $A$ is two, this implies that $\dim \ker l = \dim \mathbb{R}^3 - \text{rank } A = 1$. We may easily guess a solution of the homogeneous system associated to the last matrix, setting $z = 1$, from the second row we get $y = 1$, and from the first row, we get $x = \frac{1}{2}$. We conclude that:

$$
\ker l = \text{span}\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\} = \text{span}\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\}, \quad \text{and a basis of } \ker l \quad \text{is } B_K = \{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\}.
$$

From the Theorem on dimensions we know that $\dim \mathbb{R}^3 = \dim \ker l + \dim \text{Im } l$, from which it follows that $\dim \text{Im } l = 2 = \dim P^1$. This implies that $l$ is surjective.

5.13) Given the matrix $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, we define the linear transformation $l : \mathcal{M}^{2, 2} \to \mathcal{M}^{2, 2}$ by $l(X) = MX$.

Is $l$ an isomorphism? If possible, find its inverse.

Solution. We consider the standard basis of $\mathcal{M}^{2, 2}$: $C = \{E_1, E_2, E_3, E_4\} = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$.

We are looking for $A = A(l, C, C)$, the matrix associated to the linear transformation $l$ with respect to the basis $C$. We will follow the approach described at the beginning of Solution of Exercise 3, and evaluate:

$$
l(E_1) = M \cdot E_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 1E_1 + 0E_2 + 3E_3 + 0E_4,
$$
$$
l(E_2) = M \cdot E_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0E_1 + 1E_2 + 0E_3 + 3E_4,
$$
$$
l(E_3) = M \cdot E_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = 2E_1 + 0E_2 + 4E_3 + 0E_4,
$$
$$
l(E_4) = M \cdot E_4 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 0E_1 + 2E_2 + 0E_3 + 4E_4.
$$

From this, it follows that:

$$
A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}.
$$

To determine if $l$ is an isomorphism we will check if $\det A \neq 0$, indeed $l$ is invertible if and only if $A$ is invertible, and in this case the matrix associated to $l^{-1}$ would be $B = B(l^{-1}, C, C) = A^{-1}$. We evaluate the determinant of $A$ on the first row:

$$
\det A = 1 \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 3 & 0 & 4 \end{pmatrix} + 0 + 2 \det \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 0 \\ 0 & 3 & 4 \end{pmatrix} = -8 + 12 = 4 \neq 0.
$$
Thus, $l$ is an isomorphism and is invertible. We can directly evaluate the inverse of $l$ observing that from $Y = l(X) = M \cdot X$, and the fact that $M$ is invertible because $\det M = 4 - 6 = -2 \neq 0$, it follows that $X = l^{-1}(Y) = M^{-1}Y$. We immediately evaluate $M^{-1} = \frac{1}{\det M} \text{adj} M = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$, and conclude that $l^{-1} : M^{2,2} \rightarrow M^{2,2}$ is defined as

$$l^{-1}(Y) = M^{-1}Y = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \cdot Y.$$ 

**Observation.** In general, after we establish that a linear transformation is invertible, we may evaluate the matrix associated to the linear transformation $l^{-1}$ recalling that this is equal to $A^{-1}$. For the sake of solving the exercise in a second way, we will show also this method. We use the formula $A^{-1} = \frac{1}{\det A} \text{adj} A$. We will thus evaluate all cofactors of the given matrix.

$$A_{11} = (-1)^{1+1} \cdot \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 3 & 0 & 4 \end{pmatrix} = -8$$

$$A_{12} = (-1)^{1+2} \cdot \det \begin{pmatrix} 0 & 0 & 2 \\ 3 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 0$$

$$A_{13} = (-1)^{1+3} \cdot \det \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 0 \\ 0 & 3 & 4 \end{pmatrix} = 6$$

$$A_{14} = (-1)^{1+4} \cdot \det \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 4 \\ 0 & 3 & 0 \end{pmatrix} = 0$$

$$A_{21} = (-1)^{2+1} \cdot \det \begin{pmatrix} 0 & 2 & 0 \\ 0 & 4 & 0 \\ 3 & 0 & 4 \end{pmatrix} = 0$$

$$A_{22} = (-1)^{2+2} \cdot \det \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = -8$$

$$A_{23} = (-1)^{2+3} \cdot \det \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 4 \end{pmatrix} = 0$$

$$A_{24} = (-1)^{2+4} \cdot \det \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 4 \\ 0 & 3 & 0 \end{pmatrix} = 6$$

$$A_{31} = (-1)^{3+1} \cdot \det \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 3 & 0 & 4 \end{pmatrix} = 4$$

$$A_{32} = (-1)^{3+2} \cdot \det \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{pmatrix} = 0$$

$$A_{33} = (-1)^{3+3} \cdot \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix} = -2$$

$$A_{34} = (-1)^{3+4} \cdot \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix} = 0$$

$$A_{41} = (-1)^{4+1} \cdot \det \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 4 & 0 \end{pmatrix} = 0$$
Solution.

We recall that, given two linear transformations

$\begin{align*}
A_{42} &= (-1)^{4+2} \cdot \det \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 3 & 4 & 0 \end{pmatrix} = 4 \\
A_{43} &= (-1)^{4+3} \cdot \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 3 & 0 & 0 \end{pmatrix} = 0 \\
A_{44} &= (-1)^{4+4} \cdot \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix} = -2.
\end{align*}$

Now we can write $A^{-1} = \frac{1}{4} \begin{pmatrix} -8 & 0 & 6 \\ 0 & -8 & 0 \\ 4 & 0 & -2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4 & 0 & 2 \\ 0 & -4 & 0 \\ 3 & 0 & -1 \end{pmatrix}$. We conclude that, for every $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}^{2 \times 2}$:

$\begin{align*}
[l(Y)]_C &= A^{-1}[Y]_C = \frac{1}{2} \begin{pmatrix} -4 & 0 & 2 \\ 0 & -4 & 0 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4a + 2c \\ -4b + 2d \\ 3a - 2c \end{pmatrix}.
\end{align*}$

From this it follows that:

$\begin{align*}
l(Y) &= \frac{1}{2} \begin{pmatrix} -4a + 2c & -4b + 2d \\ 3a - 2c & 3b - 2d \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \cdot Y = M^{-1}Y.
\end{align*}$

**5.15)** Given the linear transformations

$f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (2x - y, 2y + z)$, and $g : \mathbb{R}^2 \to \mathbb{R}^3$, $g(u, v) = (u, u + v, u - v)$,

find the matrix associated to $f \circ g$ and $g \circ f$ with respect to the standard basis. Find rank($f \circ g$) and rank($g \circ f$), is one of the two compositions an isomorphism? If yes find its inverse, otherwise identify Kernel and Image.

**Solution.** We recall that, given two linear transformations $l : L \to M$, $h : M \to N$, their composition $h \circ l : L \to N$ is linear, moreover, if we fix bases $B_L, B_M, B_N$ of the linear spaces $L, M, N$, then the matrices associated to the linear transformations are such that:

$A_{h \circ l} = A(h \circ l, B_L, B_N) = A(h, B_M, B_N) \cdot A(l, B_L, B_M) = A_h \cdot A_l.$

We will thus work with the matrices associated to the given linear transformations $f, g$ with respect to the canonical bases $B_3$ and $B_2$ of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively: $B_3 = \{(1, 0), (0, 1)\}$, $B_2 = \{(1, 0), (0, 1), (0, 0, 1)\}$. Following the approach explained in Solution of Exercise 3, we evaluate:

$f(1, 0, 0) = (2, 0)$, \hspace{1cm} $f(0, 1, 0) = (-1, 2)$ \hspace{1cm} $f(0, 0, 1) = (0, 1)$.

It follows that

$A_f = A(f, B_3, B_2) = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$

Similarly, for the linear function $g$ we have:

$g(1, 0) = (1, 1, 1)$, \hspace{1cm} $g(0, 1) = (0, 1, -1)$.

It follows that

$A_g = A(g, B_2, B_3) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}.$

We are now ready to evaluate the matrix associated to $f \circ g : \mathbb{R}^2 \to \mathbb{R}^2$:

$A_{f \circ g} = A_f \cdot A_g = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}.$
From this we have the explicit definition of $f \circ g$:

$$(f \circ g)(u, v) = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (u - v, 3u + v).$$

Similarly, we evaluate the matrix associated to $g \circ f : \mathbb{R}^3 \to \mathbb{R}^3$:

$$A_{gof} = A_g \cdot A_f = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}.$$

Thus, the explicit definition of $g \circ f$ is:

$$(g \circ f)(x, y, z) = \begin{pmatrix} 2 & -1 & 0 \\ 2 & 1 & 1 \\ 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (2x - y, 2x + y + z, 2x - 3y - z).$$

By definition, the rank of a linear transformation is the dimension of its Image. In order to find $\dim \text{Im}(f \circ g)$ and $\dim \text{Im}(g \circ f)$, we will first evaluate the determinants of the matrices that represent these linear transformations, because, in case of non-zero determinants, the linear transformations are isomorphisms and the dimension of their Images is maximal.

$$\det A_{fog} = \det \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} = 4 \neq 0,$$

$$\det A_{gof} = \det \begin{pmatrix} 2 & -1 & 0 \\ 2 & 1 & 1 \\ 2 & -3 & -1 \end{pmatrix} = -2 - 2 + 6 - 2 = 0.$$

Thus, $f \circ g$ is an isomorphism, $\text{Ker}(f \circ g) = \{(0,0)\}$, $\text{Im}(f \circ g) = \mathbb{R}^2$, $\text{rank}(f \circ g) = 2$. $f \circ g$ is invertible and the matrix associated to $(f \circ g)^{-1}$ is $A_{(fog)^{-1}} = A_{fog}^{-1}$, that can be immediately evaluated using the formula $A^{-1} = \frac{1}{\det A} \text{adj} A$:

$$A_{(fog)^{-1}} = A_{fog}^{-1} = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}.$$

From this, we can write the explicit definition of $(f \circ g)^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$:

$$(f \circ g)^{-1}(u, v) = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u/4 + v/4, -u/4 + v/4 \end{pmatrix}.$$

The composition $g \circ f$ is not an isomorphism. Let’s identify its Kernel,

$$\text{Ker}(g \circ f) = \{(x, y, z) : (g \circ f)(x, y, z) = (0, 0, 0)\} = \{(x, y, z) : A_{(gof)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\}.$$

this is the space of solutions of the homogeneous system associated to $A_{(gof)}$. We will use the matrix form of Gaussian reduction and get:

$$\begin{pmatrix} 2 & -1 & 0 \\ 2 & 1 & 1 \\ 2 & -3 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have that $\text{rank} A_{(gof)} = \text{rank}(g \circ f) = 2$, this implies that $\dim \text{Ker}(g \circ f) = \dim \mathbb{R}^3 - \text{rank} A = 1$. We may easily guess a solution of the homogeneous system associated to the last matrix, setting $y = 1$, from the second row we get $z = -2$, and from the first row, we get $x = \frac{1}{2}$. We conclude that:

$$\text{Ker}(g \circ f) = \text{span}\{\frac{1}{2}, 1, -2\} = \text{span}\{1, 2, -4\}, \text{ and a basis of } \text{Ker}(g \circ f) \text{ is } B_K = \{(1, 2, -4)\}.$$

From the Theorem on dimensions we get again that $\dim \mathbb{R}^3 = \dim \text{Ker}(g \circ f) + \dim \text{Im}(g \circ f)$, from which it follows that $\dim \text{Im}(g \circ f) = 2$, as we already observed.

$$\text{Im}(g \circ f) = \text{span}\{(g \circ f)(1, 0, 0), (g \circ f)(0, 1, 0), (g \circ f)(0, 0, 1)\} = \text{span}\{(2, 2, 2), (-1, 1, -3), (0, 1, -1)\} = \text{span}\{(-1, 1, -3), (0, 1, -1)\},$$
where the last two vectors were chosen randomly, just considering that, because they are linearly independent (they are not a multiple of each other), they must span the space \( \text{Im}(g \circ f) \) that has dimension 2. Thus, a basis of \( \text{Im}(g \circ f) \) is \( B(g \circ f) = \{(−1, 1, −3), (0, 1, −1)\} \).

**5.17)** For the space \( \mathbb{R}^3 \) are given two ordered bases:

\[
B = \{(1, 0, 0), (1, 2, 0), (1, 2, 1)\}, \quad S = \{(2, -2, 2), (2, -1, -2), (0, 2, -2)\}.
\]

Find \( P_{B \rightarrow S} \) the matrix of transition from basis \( B \) to basis \( S \).

(Hint: use \( C \), the standard basis of \( \mathbb{R}^3 \), and the fact that \( P_{B \rightarrow S} = P_{S \rightarrow C}^{-1} P_{B \rightarrow C} \).)

**Solution.** The matrix of transition from a basis \( B \) to a basis \( S \), of the same linear space \( L \), is a matrix \( P_{B \rightarrow S} \) with the property that, given any vector \( v \in L \):

\[
P_{B \rightarrow S}[v]_B = [v]_S,
\]

where \([v]_X\) are the coordinates of the vector \( v \) with respect to the bases \( X \) (seen as a column). Thus, the matrix \( P_{B \rightarrow S} = A(Id, B, S) \) can be seen as the matrix associated to the linear transformation \( Id : (L, B) \rightarrow (L, S) \), where \( Id(v) = v \) is the identity map.

Let’s start considering the map \( Id : (\mathbb{R}^3, B) \rightarrow (\mathbb{R}^3, C) \), where \( C \) is the standard basis of \( \mathbb{R}^3 \), \( C = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \). To evaluate \( P_{B \rightarrow C} = A(Id, B, C) \), following the method introduced in Solution of Exercise 3, we calculate

\[
Id(1,0,0) = (1,0,0) \quad Id(1,2,0) = (1,2,0) \quad Id(1,2,1) = (1,2,1),
\]

thus \( P_{B \rightarrow C} = A(Id, B, C) = \begin{pmatrix}
1 & 1 & 1 \\
0 & 2 & 2 \\
0 & 0 & 1
\end{pmatrix} \).

Similarly, considering the map \( Id : (\mathbb{R}^3, S) \rightarrow (\mathbb{R}^3, C) \), we get \( P_{S \rightarrow C} = A(Id, S, C) = \begin{pmatrix}
2 & 2 & 0 \\
-2 & -1 & -2 \\
0 & 0 & -2
\end{pmatrix} \).

The Identity map is obviously an isomorphism, thus invertible, in particular the map \( Id : (\mathbb{R}^3, B) \rightarrow (\mathbb{R}^3, C) \) has as inverse the map \( Id : (\mathbb{R}^3, C) \rightarrow (\mathbb{R}^3, S) \). For the theorem on matrices associated to inverse maps we have \( A(Id, C, S) = A(Id, S, C)^{-1} \), thus \( P_{C \rightarrow S} = P_{S \rightarrow C}^{-1} \).

Consider the following diagram:

\[
\begin{array}{ccc}
\mathbb{R}^3 \rightarrow & (\mathbb{R}^3, S) \\
Id & \downarrow & Id \uparrow \\
(\mathbb{R}^3, C) & \rightarrow & (\mathbb{R}^3, B)
\end{array}
\]

illustrating how the map \( Id : (\mathbb{R}^3, B) \rightarrow (\mathbb{R}^3, S) \) can be obtained as a composition of two Identity maps:

\[
Id : (\mathbb{R}^3, B) \rightarrow (\mathbb{R}^3, S) = [Id : (\mathbb{R}^3, C) \rightarrow (\mathbb{R}^3, S)] \circ [Id : (\mathbb{R}^3, B) \rightarrow (\mathbb{R}^3, C)].
\]

From this, and our previous considerations, we get:

\[
A(Id, B, S) = A(Id, C, S) \cdot A(Id, B, C),
\]

or, equivalently:

\[
P_{B \rightarrow S} = P_{C \rightarrow S} P_{B \rightarrow C} = P_{S \rightarrow C}^{-1} P_{B \rightarrow C}.
\]

We need to evaluate \( P_{S \rightarrow C}^{-1} = \begin{pmatrix}
2 & 2 & 0 \\
-2 & -1 & -2 \\
0 & 0 & -2
\end{pmatrix}^{-1} \). Let us indicate \( P_{S \rightarrow C} = P \). Since \( P^{-1} = \frac{1}{\det P} \text{adj } P \), we calculate: \( \det P = \det \begin{pmatrix}
2 & 2 & 0 \\
-2 & -1 & -2 \\
0 & 0 & -2
\end{pmatrix} = -4 \), and then we proceed evaluating all cofactors of \( P \):

\[
P_{11} = (-1)^{1+1} \cdot \det \begin{pmatrix}
-1 & -2 \\
0 & -2
\end{pmatrix} = 2
\]

\[
P_{12} = (-1)^{1+2} \cdot \det \begin{pmatrix}
-2 & -2 \\
0 & -2
\end{pmatrix} = -4
\]

55
\[
P_{13} = (-1)^{1+3} \cdot \det \begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix} = 0
\]
\[
P_{21} = (-1)^{2+1} \cdot \det \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = 4
\]
\[
P_{22} = (-1)^{2+2} \cdot \det \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = -4
\]
\[
P_{23} = (-1)^{2+3} \cdot \det \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} = 0
\]
\[
P_{31} = (-1)^{3+1} \cdot \det \begin{pmatrix} 2 & 0 \\ -1 & -2 \end{pmatrix} = -4
\]
\[
P_{32} = (-1)^{3+2} \cdot \det \begin{pmatrix} 2 & 0 \\ -2 & -2 \end{pmatrix} = 4
\]
\[
P_{33} = (-1)^{3+3} \cdot \det \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix} = 2.
\]

Now we can write \( P^{-1} = -\frac{1}{4} \begin{pmatrix} 2 & -4 & 0 \\ 4 & -4 & 0 \\ -4 & 4 & 2 \end{pmatrix}^T = -\frac{1}{4} \begin{pmatrix} 2 & 4 & -4 \\ -4 & -4 & 4 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & -2 & 2 \\ 2 & 2 & -2 \\ 0 & 0 & -1 \end{pmatrix}. \)

We thus evaluate \( A(Id, B, S) = P_{B \to S}, \) calculating the indicated product:

\[
P_{B \to S} = A(Id, B, S) = A(Id, C, S) \cdot A(Id, B, C) = \frac{1}{2} \begin{pmatrix} -1 & -2 & 2 \\ 2 & 2 & -2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & -5 & -3 \\ 2 & 6 & 4 \\ 0 & 0 & -1 \end{pmatrix}.
\]

5.19) Let \( A(l) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \) be the matrix associated to a linear transformation \( l : \mathbb{R}^3 \to \mathbb{R}^2 \) with respect to the standard bases of \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \). Find the matrix associated to the given transformation with respect to the bases \( S \) and \( B \), where:

\[
S = \{(2, -2, 2), (2, -1, -2), (0, 2, -2)\}, \quad \text{and} \quad B = \{(1, 1), (1, -1)\}.
\]

**Solution.** The given \( A(l) = A(l, C_3, C_2) \) is the matrix associated to the linear transformation \( l : (\mathbb{R}^3, C_3) \to (\mathbb{R}^2, C_2) \) with respect to the canonical bases \( C_3, C_2 \) of \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \) respectively. We are asked to find \( A(l, S, B) \).

Let’s consider the following diagram:

\[
\begin{array}{ccc}
(\mathbb{R}^3, S) & \overset{l}{\longrightarrow} & (\mathbb{R}^2, B) \\
Id & \downarrow & Id \\
(\mathbb{R}^3, C_3) & \overset{l}{\longrightarrow} & (\mathbb{R}^2, C_2)
\end{array}
\]

where \( Id \) indicates the Identity map. This illustrates how the map \( l : (\mathbb{R}^3, S) \to (\mathbb{R}^2, B) \) can be obtained as a composition of \( Id : (\mathbb{R}^3, S) \to (\mathbb{R}^3, C_3), l : (\mathbb{R}^3, C_3) \to (\mathbb{R}^2, C_2), \) and \( Id : (\mathbb{R}^2, C_2) \to (\mathbb{R}^2, B) \):

\[
l : (\mathbb{R}^3, S) \to (\mathbb{R}^2, B) = [Id : (\mathbb{R}^2, C_2) \to (\mathbb{R}^2, B)] \circ [l : (\mathbb{R}^3, C_3) \to (\mathbb{R}^2, C_2)] \circ [Id : (\mathbb{R}^3, S) \to (\mathbb{R}^3, C_3)],
\]

or, in terms of the matrices associated to the linear transformations:

\[
A(l, B, S) = A(Id, C_2, B) \cdot A(l, C_3, C_2) \cdot A(Id, S, C_3).
\]

Using the notation and the notions introduced in Solution of Exercise 17:

\[
A(l, B, S) = P_{C_2 \to B} \cdot A(l) \cdot P_{S \to C_3} = P_{B \to C_2}^{-1} \cdot A(l) \cdot P_{S \to C_3},
\]

where \( A(l) \) is the given matrix, and, using the considerations explained in Solution of Exercise 17:

\[
P_{S \to C_3} = \begin{pmatrix} 2 & 2 & 0 \\ -2 & -1 & 2 \\ 2 & -2 & -2 \end{pmatrix} \quad \text{and} \quad P_{B \to C_2} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]
We immediately evaluate \( P_{B \to C_2} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \). Now, we get \( A(l, B, S) \) calculating the indicated product:

\[
A(l, B, S) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ -2 & -1 & 2 \\ 2 & -2 & -2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & 0 \\ 8 & -1 & -6 \end{pmatrix} \begin{pmatrix} 10 & -2 & -6 \\ -6 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 5 & -1 & -3 \end{pmatrix}.
\]

5.21) Given the linear transformation \( l_h : \mathbb{R}^3 \to \mathbb{R}^3, l_h(x, y, z) = (2x + y, y - z, 2y + hz) \), write the matrix associated to \( l_h \) with respect to the standard basis of \( \mathbb{R}^3 \), find \( \text{Ker} l, \text{Im} l \), its bases and dimensions for every possible value of \( h \in \mathbb{R} \). Does there exist a value of \( h \) for which \( l_h \) is an isomorphism?

**Solution.** We consider the standard basis \( C \) of \( \mathbb{R}^3, C = \{(1, 0, 0), (0, 1, 1), (0, 0, 1)\} \), and, following the method explained in Solution of Exercise 3, we evaluate \( A_h = A_h(l_h, C, C) \) as follows:

\[
l(1, 0, 0) = (2, 0, 0), \quad l(0, 1, 0) = (1, 1, 2), \quad l(0, 0, 1) = (0, -1, h),
\]

from which we deduce:

\[
A_h = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & h \end{pmatrix}.
\]

To find \( \text{Ker} l_h = \{ (x, y, z) \in \mathbb{R}^3 : A_h \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} \), the space of solutions of the homogeneous system associated to \( A_h \), we first evaluate \( \det A_h \), because, for all values of \( h \in \mathbb{R} \) such that \( \det A_h \neq 0 \), we know that the homogeneous system has a unique solution, the trivial one, thus \( \text{Ker} l_h = \{0\} \).

Since \( \det A_h = 2h + 4 = 2(h + 2) \), we conclude that, for \( h \neq -2 \), \( \text{Ker} l_h = \{(0, 0, 0)\} \), consequently \( \text{Im} l_h = \mathbb{R}^3 \), that means that \( l_h \) is an isomorphism.

For \( h = -2 \), we need to find a basis of

\[
\text{Ker} l_2 = \{ (x, y, z) \in \mathbb{R}^3 : A_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} = \{ (x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \},
\]

that means, we need to solve the homogeneous system:

\[
\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} R_3 - 2R_2 \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Setting \( z = \alpha, \alpha \in \mathbb{R} \), we get from the second row (imagining the correspondent equation \( y - z = 0 \) \( y = \alpha \)), and, subsequently, from the first row \( x = -\frac{1}{2} \alpha \). Thus we conclude that:

\( \text{Ker} l_2 = \{(\alpha, \alpha, -\frac{1}{2} \alpha) : \alpha \in \mathbb{R}\} = \text{span}\{(1, 1, -\frac{1}{2})\} = \text{span}\{(2, 2, -1)\}, \dim \text{Ker} l_2 = 1 \), and a basis of \( \text{Ker} l_2 \) is \( B_{\text{K}2} = \{(2, 2, -1)\} \).

From the Theorem on dimensions (ii) Solution of Exercise 3), we get \( \dim \text{Im} l_2 = \dim \mathbb{R}^3 - \dim \text{Ker} l_2 = 2 \), and, from Theorem i) we get

\( \text{Im} l_2 = \text{span}\{l((1, 0, 0), (0, 1, 0), (0, 0, 1)) = \text{span}\{(2, 0, 0), (1, 1, 2), (0, -1, -2)\} = \text{span}\{(2, 0, 0), (1, 1, 2)\} \),

where the first two vectors were chosen because they are linearly independent (they are not a multiple of each other), thus they must span the space \( \text{Im} l_2 \) that has dimension 2. A basis of \( \text{Im} l_2 \) is \( B_{\text{I}} = \{(2, 0, 0), (1, 1, 2)\} \).
6.1) Determine if \( A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \) and \( C = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \) are diagonalizable (in \( \mathbb{R} \)).

6.2) Determine if \( A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \) and \( B = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix} \) are diagonalizable.

Notice that the given matrices have the same characteristic polynomial but they are not similar.

6.3) Diagonalize the following matrices (for each eigenvalue find a basis of the corresponding eigenspace)

\[
A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.
\]

6.4) Diagonalize the following matrices (for each eigenvalue find a basis of the corresponding eigenspace)

\[
A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -2 \end{pmatrix}.
\]

6.5) Are the following matrices diagonalizable in \( \mathbb{R} \)?

\[
A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.
\]

6.6) Are the following matrices diagonalizable in \( \mathbb{R} \)?

\[
A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 2 & -2 \\ 2 & 5 & -2 \\ -2 & -2 & 5 \end{pmatrix}.
\]

6.7) Given the transformation \( l : \mathbb{R}^3 \to \mathbb{R}^2, l(x, y, z) = (2x + y, y - z, 2y + 4z) \), find all eigenvalues and a basis of each eigenspace. Is \( l \) diagonalizable? Determine if \( l \) is invertible.

6.8) Given the linear transformations from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) defined below, find characteristic equation, real eigenvalues and corresponding eigenvectors.

a) \( l(x, y) = (x + 5y, 2x + 4y) \),

b) \( f(1, 0) = (2, -5) \), \( f(0, 1) = (4, 4) \).

9) Given the linear transformation \( l : \mathbb{R}^2 \to \mathbb{R}^2, l(x, y) = (x - y, x + 3y) \), write the matrix \( A(l, S, S) \) associated to \( l \) with respect to the basis \( S = \{(1, 2), (2, -2)\} \). Determine if there exists a basis \( B \) of \( \mathbb{R}^2 \) such that \( A(l, B, B) \) is diagonal.

10) Given the linear transformation \( l : \mathbb{R}^3 \to \mathbb{R}^3, l(x, y, z) = (2x + y + 3z, x + 3y + z, x) \) find a basis \( B \) of eigenvectors of \( l \), such that the matrix \( D \) associated to \( l \) with respect to \( B \) is diagonal. Verify that \( P^{-1}AP = D \), where \( P \) is the transition matrix from basis \( B \) to the canonical one, and \( A \) is the matrix associated to \( l \) with respect to the canonical basis.

11) Given the transformation \( f_h : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by \( f_h(x, y, z) = (x - hz, x + y - hz, -hx + z) \), where \( h \in \mathbb{R} \) is a parameter, determine if \( f_h \) is diagonalizable for some values of \( h \). For such values of \( h \), find a basis of \( \mathbb{R}^3 \) of eigenvectors of \( f_h \).

12) Given the linear transformation \( l : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by
\[ l(x, y, z) = (5x + 6y - 3z, 6x + 9y, -3x + 9z), \]

find a diagonal matrix associated to \( l \) (if it exists) and the corresponding basis of \( \mathbb{R}^3 \).

6.13) Find the linear transformation \( l : \mathbb{R}^3 \to \mathbb{R}^3 \) stretching every vector on the plane \( x + 2y + 3z = 0 \) twice and reversing the vector \((0, 1, 1)\).

6.14) Find the linear transformation \( l : \mathbb{R}^3 \to \mathbb{R}^3 \) stretching twice the vector \((1, 0, 0)\), three times the vector \((1, 1, 0)\) and mapping \((1, 0, 1)\) to \((0, 0, 0)\).

6.15) Find the linear transformation \( l : \mathbb{R}^2 \to \mathbb{R}^2 \) knowing that \((1, 2)\) and \((1, 3)\) are two of its eigenvectors and \(l(1,0) = (4,6)\).

6.16) Find the linear transformation \( l : \mathbb{R}^2 \to \mathbb{R}^2 \) knowing that \((2, 2)\) and \((-1, 3)\) are two of its eigenvectors and \(l(0,1) = (2,1)\).

6.17) Given \( A = \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \), use the eigenvalue method

a) to derive an explicit formula for \( A^n \),

b) to solve the system of differential equations

\[
\begin{align*}
\frac{dx}{dt} &= 2x - 3y \\
\frac{dy}{dt} &= 4x - 5y.
\end{align*}
\]

6.18) Given \( A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \), use the eigenvalue method

a) to derive an explicit formula for \( A^n \),

b) to solve the system of differential equations

\[
\begin{align*}
\frac{dx}{dt} &= 2x + y \\
\frac{dy}{dt} &= x + 2y.
\end{align*}
\]

6.19) Solve the system of recurrence relations

\[
\begin{align*}
x_{n+1} &= 2x_n - y_n \\
y_{n+1} &= -x_n + 2y_n
\end{align*}
\]

given that \(x_0 = 0\) and \(y_0 = -1\).

6.20) Solve the system of recurrence relations

\[
\begin{align*}
x_{n+1} &= 3x_n - y_n \\
y_{n+1} &= -x_n + 3y_n
\end{align*}
\]

given that \(x_0 = 1\) and \(y_0 = 2\).
Solutions of odd number problems. Eigenvalues, eigenvectors and their applications

**Basic facts:** Given a matrix \( A \in \mathcal{M}^{n \times n} \), a real number \( \lambda \in \mathbb{R} \) is called an eigenvalue of \( A \) if there exists a nontrivial vector \( v \in \mathbb{R}^n \) (\( v \neq 0 \)), such that \( Av = \lambda v \). The vector \( v \) is said to be an eigenvector of \( A \), associated to the eigenvalue \( \lambda \). The set of all eigenvectors associated to the same eigenvalue \( \lambda \), plus the zero vector, form a linear subspace of \( \mathbb{R}^n \), called eigenspace associated to the eigenvalue \( \lambda \). Eigenvectors corresponding to different eigenvalues are linearly independent. If there exists a basis of \( \mathbb{R}^n \) made of eigenvectors of \( A \), then \( A \) is diagonalizable and, if \( P \) is the matrix formed with the \( n \) linearly independent eigenvectors of \( A \) written as columns, then \( P^{-1}AP = D \), where \( D \) is a diagonal matrix, with the only nonzero entries in the diagonal formed by the corresponding eigenvalues of \( A \).

In order to identify the eigenvalues of a matrix \( A \), we observe that, looking for a nontrivial solution of the equation \( Av = \lambda v \) is equivalent to finding nontrivial solutions of the homogeneous system \((A - \lambda I)v = 0\), thus we must find the possible values of \( \lambda \in \mathbb{R} \) such that \( \det(A - \lambda I) = 0 \). The last equation is called characteristic equation of \( A \). Given one solution \( \lambda \) of the characteristic equation, the eigenvectors corresponding to \( \lambda \) are all nontrivial solutions of the system \((A - \lambda I)v = 0\).

6.1) Determine if \( A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \) and \( C = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \) are diagonalizable (in \( \mathbb{R} \)).

**Solution.** Let’s consider \( A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \), we start by finding the real solutions of \( \det(A - \lambda I) = 0 \). In this case, we have:

\[
\det \left( \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \det \begin{pmatrix} 3 - \lambda & -1 \\ 1 & 1 - \lambda \end{pmatrix} = (3 - \lambda)(1 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.
\]

The characteristic equation of \( A \) has a unique solution \( \lambda = 2 \) with algebraic multiplicity 2. We now look for the eigenvectors corresponding to \( \lambda = 2 \). We must find the nontrivial solutions of the homogeneous system \((A - \lambda I)v = 0\), where \( \lambda = 2 \). Thus, we look for \( v = (x, y) \neq (0, 0) \) so that

\[
\begin{pmatrix} 3 - 2 & -1 \\ 1 & 1 - 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

We solve the homogeneous system with augmented matrix:

\[
\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} R_2 - R_1 \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

All the eigenvectors corresponding to \( \lambda = 2 \) are \{t(1, 1) : t \in \mathbb{R} \setminus \{0\}\} and the eigenspace associated to \( \lambda = 2 \) is \( U_2 = \text{span}\{(1, 1)\} \). We found only one linearly independent eigenvector corresponding to the eigenvalue with algebraic multiplicity two, thus the given matrix \( A \) is not diagonalizable.

Let’s now consider \( B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \). We solve \( \det(B - \lambda I) = 0 \).

\[
\det \left( \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \det \begin{pmatrix} 1 - \lambda & -1 \\ 2 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^2 + 2.
\]

The characteristic equation of \( B \) has no real solutions, thus we did not find any real eigenvalue, the matrix is not diagonalizable on the reals.

Let’s now consider \( C = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \). We solve \( \det(C - \lambda I) = 0 \).

\[
\det \left( \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \det \begin{pmatrix} 2 - \lambda & 1 & 1 \\ 0 & 3 - \lambda & 1 \\ 0 & 0 & 2 - \lambda \end{pmatrix} = (2 - \lambda)^2(3 - \lambda).
\]

The characteristic equation has solutions \( \lambda = 2 \) with algebraic multiplicity two, and \( \lambda = 3 \) with algebraic multiplicity one. We observe that \( C \) is an upper triangular matrix, thus the eigenvalues are exactly the
numbers on the diagonal and their multiplicity corresponds with the number of times they appear on the diagonal.

We now look for the eigenvectors corresponding to \( \lambda = 2 \). We must find the nontrivial solutions of the homogeneous system \((C - \lambda I)v = 0\), where \( \lambda = 2 \). The augmented matrix of the system is

\[
\begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
R_2 - R_1 \sim
\begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

All the eigenvectors corresponding to \( \lambda = 2 \) are \( \{(t, 0, 1, -1) + u(1, 0, 0) : t, u \in \mathbb{R} \text{ non both zero}\} \), and the eigenspace corresponding to \( \lambda = 2 \) is \( U_2 = \text{span}\{(1, 0, 0), (0, 1, -1)\}\).

We now look for the eigenvectors corresponding to \( \lambda = 3 \). We must find the nontrivial solutions of the homogeneous system \((C - \lambda I)v = 0\), where \( \lambda = 3 \). The augmented matrix of the system is

\[
\begin{pmatrix}
-1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}
R_1 - R_2 \sim
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

All the eigenvectors corresponding to \( \lambda = 3 \) are \( \{s(1, 1, 0) : s \in \mathbb{R} \setminus \{0\}\} \), and the eigenspace corresponding to \( \lambda = 3 \) is \( U_3 = \text{span}\{(1, 1, 0)\} \). Thus the set \( B = \{(1, 0, 0), (0, 1, -1), (1, 1, 0)\} \) is basis of \( \mathbb{R}^3 \), formed by 3 linearly independent eigenvectors of \( C \), and \( C \) is diagonalizable. One matrix \( P \) that diagonalizes \( C \) is:

\[
P = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & -1 & 0
\end{pmatrix},
\]

and in this case:

\[
P^{-1}CP = D = \begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}.
\]

By changing the order of the vectors of \( B \) as columns of \( P \), we may find several different other matrices that diagonalize \( C \), for example:

\[
P' = \begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & -1 & 0
\end{pmatrix},
\]

in this case:

\[
P'^{-1}CP' = D' = \begin{pmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

6.3) Diagonalize the following matrices (for each eigenvalue find a basis of the corresponding eigenspace):

\[
A = \begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 0 \\
-1 & 1 & 1
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
0 & 1 & 4 \\
1 & -1 & 1 \\
1 & 0 & 1
\end{pmatrix}.
\]

Solution. Let’s consider \( A = \begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 0 \\
-1 & 1 & 1
\end{pmatrix} \). We start by finding the real solutions of \( \det(A - \lambda I) = 0 \). In this case, we have:

\[
\det \begin{pmatrix}
-2 - \lambda & 1 & 0 \\
1 & -2 - \lambda & 0 \\
-1 & 1 & 1
\end{pmatrix} = \det \begin{pmatrix}
-2 - \lambda & 1 & 0 \\
1 & -2 - \lambda & 0 \\
0 & 0 & 1
\end{pmatrix} = (1 - \lambda)(3 + \lambda)(1 + \lambda).
\]

The characteristic equation of \( A \) has three distinct solutions \( \lambda = 1 \), \( \lambda = -3 \), \( \lambda = -1 \), with algebraic multiplicity 1. In this case, we may already conclude that \( A \) is diagonalizable, indeed, corresponding to each eigenvalue we will choose an eigenvector, and, since the eigenvalues are distinct, the chosen eigenvectors are linearly independent and form a basis of \( \mathbb{R}^3 \).

We now look for the eigenvectors corresponding to \( \lambda = 1 \). We must find the nontrivial solutions of the homogeneous system \((A - \lambda I)v = 0\), where \( \lambda = 1 \). Thus, we solve the homogeneous system with augmented matrix:

\[
\begin{pmatrix}
-3 & 1 & 0 & 0 \\
1 & -3 & 0 & 0 \\
-1 & 1 & 0 & 0
\end{pmatrix}.
\]

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All the eigenvectors corresponding to $\lambda = 1$ are $\{t(0, 0, 1) : t \in \mathbb{R} \setminus \{0\}\}$ and the eigenspace associated to $\lambda = 1$ is $U_1 = \text{span}\{0, 0, 1\}$.

We consider $\lambda = -3$, and find the corresponding eigenvectors solving the system:

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
-1 & 1 & 4
\end{pmatrix}
\begin{pmatrix}
R_2 - R_1 \\
R_3 + R_1
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 2 & 4
\end{pmatrix}.
\]

It is easy to deduce that all the eigenvectors corresponding to $\lambda = -3$ are $\{t(2, -2, 1) : t \in \mathbb{R} \setminus \{0\}\}$ and the eigenspace associated to $\lambda = -3$ is $U_{-3} = \text{span}\{(2, -2, 1)\}$.

Finally, we consider the eigenvalue $\lambda = -1$, and find the corresponding eigenvectors solving the system:

\[
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
-1 & 1 & 2 & 0
\end{pmatrix}
\begin{pmatrix}
R_2 + R_1 \\
R_3 + R_2
\end{pmatrix}
\sim
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0
\end{pmatrix}.
\]

The eigenvectors corresponding to $\lambda = -1$ are $\{t(1,1,0) : t \in \mathbb{R} \setminus \{0\}\}$ and the eigenspace associated to $\lambda = -1$ is $U_{-1} = \text{span}\{(1,1,0)\}$.

We conclude that the given matrix $A$ is similar to a diagonal matrix, and the matrix $P = \begin{pmatrix} 0 & 2 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is such that

\[
P^{-1}AP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Let’s consider $C = \begin{pmatrix} 0 & 1 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Looking for real solutions of $\det(C - \lambda I) = 0$, we have:

\[
\det \begin{pmatrix}
0 & 1 & 4 \\
1 & -1 & 1 \\
1 & 0 & 1
\end{pmatrix} - \lambda \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \det \begin{pmatrix}
-\lambda & 1 & 0 \\
1 & -1 - \lambda & 1 \\
1 & 0 & 1 - \lambda
\end{pmatrix} = -\lambda^3 + 6\lambda + 4.
\]

To find the solution of the characteristic equation, we must find the roots of the polynomial $p(\lambda) = -\lambda^3 + 6\lambda + 4$.

Among the integer divisors of 4, we choose $-2$, and with the use of a Horner’s schema we evaluate:

\[
\begin{array}{c|ccc}
& -1 & 0 & 6 & 4 \\
-2 & 2 & -4 & -4 \\
\hline
-1 & 2 & 0 & 0 & 0
\end{array}
\]

Thus $-2$ is a root of $p(\lambda)$ and:

\[-\lambda^3 + 6\lambda + 4 = (\lambda + 2)(-\lambda^2 + 2\lambda + 2).
\]

The remaining two roots can be found using the formula for quadratic equations

\[-\lambda^3 + 6\lambda + 4 = -(\lambda + 2)(\lambda - 1 + \sqrt{3})(\lambda - 1 - \sqrt{3}).
\]

The characteristic equation of $C$ has three distinct solutions $\lambda = -2$, $\lambda = 1 - \sqrt{3}$, $\lambda = 1 + \sqrt{3}$, with algebraic multiplicity 1. Again, we may already conclude that $C$ is diagonalizable because the eigenvalues are three distinct numbers. We now look for the corresponding eigenspaces.

For $\lambda = -2$, we solve $(C - \lambda I)v = 0$, where $\lambda = -2$. Thus, we solve the homogeneous system with augmented matrix:

\[
\begin{pmatrix}
2 & 1 & 4 \\
1 & 1 & 1 \\
1 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
R_1 - R_2 - R_3 \\
\sim
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 4 \\
0 & 0 & 0 \\
1 & 0 & 3
\end{pmatrix}.
\]

All the eigenvectors corresponding to $\lambda = -2$ are $\{t(-3, 2, 1) : t \in \mathbb{R} \setminus \{0\}\}$ and the eigenspace associated to $\lambda = -2$ is $U_{-2} = \text{span}\{(-3, 2, 1)\}$.

For $\lambda = 1 - \sqrt{3}$, we solve $(C - \lambda I)v = 0$, where $\lambda = 1 - \sqrt{3}$. Thus, we solve the homogeneous system with augmented matrix:

\[
\begin{pmatrix}
-1 + \sqrt{3} & 1 & 4 \\
1 & -2 + \sqrt{3} & 1 \\
1 & 0 & \sqrt{3}
\end{pmatrix}
\begin{pmatrix}
R_2 + R_1 \\
\sim
\end{pmatrix}
\begin{pmatrix}
-1 + \sqrt{3} & 1 & 4 \\
-4 + 3\sqrt{3} & 5 - \sqrt{3} & 0 \\
0 & 0 & 2 - \sqrt{3} & \sqrt{3} - 1 & 0
\end{pmatrix}.
\]
(1 + 2√3)R_3 - R_2 \sim \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}

R_3 \leftrightarrow R_2

\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}

\begin{pmatrix} -3 + \sqrt{3} & 1 + 3 \sqrt{3} \\ 1 & 2 + \sqrt{3} \end{pmatrix}

\begin{pmatrix} -3 + \sqrt{3} & 1 + 3 \sqrt{3} \\ 1 & 2 + \sqrt{3} \end{pmatrix}

\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 + \sqrt{3} & 0 \\ 0 & 0 & 1 + \sqrt{3} \end{pmatrix}

6.5) Are the following matrices diagonalizable in \( \mathbb{R} \)?

\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}

\begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}

Solution. We first consider the matrix \( A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix} \), and look for its eigenvalues solving the characteristic equation \( \det(A - \lambda I) = 0 \). In this case, we have:

\[ \det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 2 - \lambda & 1 & 0 \\ 1 & 3 - \lambda & -1 \\ -1 & 2 & 3 - \lambda \end{pmatrix} = -\lambda^3 + 8\lambda^2 - 22\lambda + 20. \]

To find the solution of the characteristic equation, we must find the roots of the polynomial \( p(\lambda) = -\lambda^3 + 8\lambda^2 - 22\lambda + 20 \). Among the integer divisors of 20, we choose 2, and with the use of a Horner’s schema we evaluate:

\[
\begin{array}{c|c|c|c|c|c}
\lambda & 2 & -2 & -6 & -10 & 0 \\
2 & 8 & -22 & 20 & & \\
-1 & 6 & -10 & 0 & & \\
\end{array}
\]

Thus 2 is a root of \( p(\lambda) \) and:

\[-\lambda^3 + 8\lambda^2 - 22\lambda + 20 = (\lambda - 2)(-\lambda^2 + 6\lambda - 10). \]

The polynomial \(-\lambda^2 + 6\lambda - 10\) is irreducible, thus the characteristic equation has only one real root. This implies that the given matrix \( A \) is not diagonalizable on the reals.

We now consider the matrix \( B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \), and look for its eigenvalues solving the characteristic equation \( \det(B - \lambda I) = 0 \). We have:
All the eigenvectors corresponding to $\lambda$ and $\lambda$ are $\mathbb{R}$ linearly independent eigenvectors of $B$. We conclude that the set $U = \{1, 1, 0, 0\}$, formed by $4$ linearly independent eigenvectors of $B$, and $B$ is diagonalizable. One matrix $P$ that diagonalizes $B$ is:

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$
and in this case:

\[ P^{-1}BP = D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]

We now consider the matrix \( C = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \), and evaluate its eigenvalues by solving the equation

\[ \det(C - \lambda I) = 0. \]

We have:

\[ \det \left( \begin{pmatrix} 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = \det \left( \begin{pmatrix} 2 - \lambda & 2 & 2 & 2 \\ 0 & 1 - \lambda & 1 & 1 \\ 0 & 0 & 2 - \lambda & 2 \\ 0 & 0 & 0 & 1 - \lambda \end{pmatrix} \right) = (2 - \lambda)^2(1 - \lambda)^2. \]

The characteristic equation of \( C \) has two real solutions: \( \lambda = 2 \) and \( \lambda = 1 \), both with algebraic multiplicity two. We now find the corresponding eigenspaces.

For \( \lambda = 2 \), we solve the system \((C - \lambda I)v = 0\), where \( \lambda = 2 \). Thus, we solve the homogeneous system with augmented matrix:

\[ \begin{pmatrix} 0 & 2 & 2 & 2 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} R_1 + 2R_2 \sim R_3 + 2R_4 \begin{pmatrix} 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

All the eigenvectors corresponding to \( \lambda = 2 \) are \( \{t(1,0,0,0) : t \in \mathbb{R} \setminus \{0\}\} \) and the eigenspace associated to \( \lambda = 2 \) is \( U_2 = \text{span}\{(1,0,0,0)\} \). Since the eigenvalue \( \lambda = 2 \) has algebraic multiplicity two, but the dimension of the corresponding eigenspace is one, we may already conclude that the given matrix \( C \) is not diagonalizable. We anyway determine also the eigenspace corresponding to the eigenvalue \( \lambda = 1 \). We solve the system \((C - \lambda I)v = 0\), where \( \lambda = 1 \). This homogeneous system has augmented matrix:

\[ \begin{pmatrix} 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

All the eigenvectors corresponding to \( \lambda = 1 \) are \( \{t(2,-1,0,0) : t \in \mathbb{R} \setminus \{0\}\} \), the eigenspace associated to \( \lambda = 1 \) is \( U_1 = \text{span}\{(2,-1,0,0)\} \), thus also \( U_1 \) has dimension only one. \( C \) is not diagonalizable.

6.7) Given the transformation \( l : \mathbb{R}^3 \to \mathbb{R}^3 \), \( l(x, y, z) = (2x + y, y - z, 2y + 4z) \), find all eigenvalues and a basis of each eigenspace. Is \( l \) diagonalizable? Determine if \( l \) is invertible.

**Solution.** First of all, we will find the matrix \( A = A(l, C, C) \) associated to the linear transformation \( l \) with respect to the canonical basis \( C \) of \( \mathbb{R}^3 \), \( C = \{(1,0,0),(0,1,0),(0,0,1)\} \). For this purpose, we need to evaluate:

\[
\begin{align*}
l(1,0,0) &= (2,0,0) \\
l(0,1,0) &= (1,1,2) \\
l(0,0,1) &= (0,-1,4).
\end{align*}
\]

Now we can write \( A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} \). The eigenvalues and eigenvectors of \( l \) correspond with the eigenvalues and eigenvectors of \( A \), and \( l \) is diagonalizable if and only if \( A \) is diagonalizable. We look for eigenvalues of \( A \) solving the characteristic equation \( \det(A - \lambda I) = 0 \). In this case, we have:

\[
\det \left( \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \det \left( \begin{pmatrix} 2 - \lambda & 1 & 0 \\ 0 & 1 - \lambda & -1 \\ 0 & 2 & 4 - \lambda \end{pmatrix} \right) = (2 - \lambda)(1 - \lambda)(4 - \lambda) + 2(2 - \lambda) = (2 - \lambda)(\lambda^2 - 5\lambda + 6) = (2 - \lambda)^2(3 - \lambda).
\]

The characteristic equation has solutions \( \lambda = 2 \) with algebraic multiplicity two, and \( \lambda = 3 \) with algebraic multiplicity one.
We now look for the eigenvectors corresponding to $\lambda = 2$. We must find the nontrivial solutions of the homogeneous system $(\mathbf{A} - \lambda \mathbf{I}) \mathbf{v} = \mathbf{0}$, where $\lambda = 2$. The augmented matrix of the system is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & -1 & -1 & 0 \\
0 & 2 & 2 & 0
\end{pmatrix}
- R_2
\sim
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & -2 & -1 & 0 \\
0 & 2 & 2 & 0
\end{pmatrix}
R_3 + 2R_2
\sim
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

All the eigenvectors corresponding to $\lambda = 2$ are $\{t(1,0,0) : t \in \mathbb{R} \setminus \{0\}\}$, and the eigenspace corresponding to $\lambda = 2$ is $U_2 = \text{span}\{(1,0,0)\}$. Corresponding to the eigenvalue $\lambda = 2$ with algebraic multiplicity two, we found and eigenspace $U_2$ with dimension one. This implies that $\mathbf{A}$, and thus $l$, is not diagonalizable.

We anyway look for the eigenvectors corresponding to $\lambda = 3$. We must find the nontrivial solutions of the homogeneous system $(\mathbf{A} - \lambda \mathbf{I}) \mathbf{v} = \mathbf{0}$, where $\lambda = 3$. The augmented matrix of the system is

$$
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & -2 & -1 & 0 \\
0 & 2 & 1 & 0
\end{pmatrix}
- R_2
\sim
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

All the eigenvectors corresponding to $\lambda = 3$ are $\{t(1,1,-2) : t \in \mathbb{R} \setminus \{0\}\}$, and the eigenspace corresponding to $\lambda = 3$ is $U_3 = \text{span}\{(1,1,-2)\}$.

It is left to answer the question on the existence of $l^{-1}$. Using the fact that the eigenvalues of $l$ are 2 and 3, and the value 0 is not an eigenvalue of $l$, we may conclude that $\text{Ker} \ l = \{(0,0,0)\}$ and thus $l$ is invertible.

6.9) Given the linear transformation $l : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $l(x,y) = (x-y, x+3y)$, write the matrix $\mathbf{A}(l,S,S)$ associated to $l$ with respect to the basis $S = \{(1,2),(2,-2)\}$. Determine if there exists a basis $B$ of $\mathbb{R}^2$ such that $\mathbf{A}(l,B,B)$ is diagonal.

Solution. We start by finding the matrix $\mathbf{A} = \mathbf{A}(l,C,C)$ associated to $l$ with respect to the canonical basis $C = \{(1,0),(0,1)\}$ of $\mathbb{R}^2$. For this purpose, we need to evaluate:

$$
\begin{align*}
l(1,0) &= (1,1) \\
l(0,1) &= (-1,3).
\end{align*}
$$

Thus $\mathbf{A} = \mathbf{A}(l,C,C) = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ is the matrix associated to the linear transformation $l : (\mathbb{R}^2,C) \rightarrow (\mathbb{R}^2,C)$.

We are asked to find $\mathbf{A}(l,S,S)$. Using the notations introduced in Exercise 17 and 19 of the chapter on Linear Transformations, we consider the following diagram:

$$
\begin{array}{ccc}
(\mathbb{R}^2,S) & \xrightarrow{l} & (\mathbb{R}^2,S) \\
\downarrow Id & & \uparrow Id \\
(\mathbb{R}^2,C) & \xrightarrow{l} & (\mathbb{R}^2,C)
\end{array}
$$

where $Id$ indicates the Identity map. This illustrates how the map $l : (\mathbb{R}^2,S) \rightarrow (\mathbb{R}^2,S)$ can be obtained as a composition of $Id : (\mathbb{R}^2,S) \rightarrow (\mathbb{R}^2,S)$, $l : (\mathbb{R}^2,C) \rightarrow (\mathbb{R}^2,C)$, and $Id : (\mathbb{R}^2,C) \rightarrow (\mathbb{R}^2,C)$:

$$
l : (\mathbb{R}^2,S) \rightarrow (\mathbb{R}^2,S) = [Id : (\mathbb{R}^2,C) \rightarrow (\mathbb{R}^2,S)] \circ [l : (\mathbb{R}^2,C) \rightarrow (\mathbb{R}^2,C)] \circ [Id : (\mathbb{R}^2,S) \rightarrow (\mathbb{R}^2,C)],
$$

or, in terms of the matrices associated to the linear transformations:

$$
\mathbf{A}(l,S,S) = \mathbf{A}(Id,C,S) \cdot \mathbf{A}(l,C,C) \cdot \mathbf{A}(Id,S,C).
$$

Using the notation and the notions introduced in Solution of Exercise 17 in the chapter of Linear Transformations:

$$
\mathbf{A}(l,S,S) = \mathbf{P}_{C \rightarrow S} \cdot \mathbf{A} \cdot \mathbf{P}_{S \rightarrow C} = \mathbf{P}_{S \rightarrow C}^{-1} \cdot \mathbf{A}(l) \cdot \mathbf{P}_{S \rightarrow C},
$$

where $\mathbf{A}$ is the matrix we have already evaluated, and $\mathbf{P}_{S \rightarrow C}$ is the transition matrix from basis $S$ to basis $C$, formed by the vectors of basis $S$ written as columns:

$$
\mathbf{P}_{S \rightarrow C} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}.
$$
We immediately evaluate \( P_{s^{-1}} = \frac{1}{2} \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \). Now, we get \( A(l, S) \) calculating the indicated product:

\[
A(l, S, S) = \frac{1}{6} \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 4 \\ 7 & -4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 12 & 0 \\ -9 & 12 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -3/2 & 2 \end{pmatrix}.
\]

The basis \( S \) does not diagonalize the linear transformation. To verify if there exists a basis \( B \) that diagonalizes \( l \), we must evaluate the eigenvalues of \( l \) (eigenvalues of \( A \)) and find a possible basis \( B \) of \( \mathbb{R}^2 \) made of eigenvectors of \( l \). We look for eigenvalues of \( A \) solving the characteristic equation \( \det(A - \lambda I) = 0 \). In this case, we have:

\[
det \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 - \lambda & -1 \\ 1 & 3 - \lambda \end{pmatrix} = (1 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.
\]

The characteristic equation has solution \( \lambda = 2 \) with algebraic multiplicity two.

We now look for the eigenvectors corresponding to \( \lambda = 2 \). We must find the nontrivial solutions of the homogeneous system \((A - \lambda I)v = 0\), where \( \lambda = 2 \). The augmented matrix of the system is

\[
\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

All the eigenvectors corresponding to \( \lambda = 2 \) are \( \{t(1, -1) : t \in \mathbb{R} \setminus \{0\} \} \), and the eigenspace corresponding to \( \lambda = 2 \) is \( U_2 = \text{span}\{(1, -1)\} \). Corresponding to the eigenvalue \( \lambda = 2 \) with algebraic multiplicity two, we found and eigenspace \( U_2 \) with dimension one. This implies that \( A \), and thus \( l \), is not diagonalizable: there exists no basis \( B \) of \( \mathbb{R}^2 \) such that \( A(l,B,B) \) is diagonal.

**6.11** Given the transformation \( f_h : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by \( f_h(x, y, z) = (x - hz, x + y - hz, -hx + z) \), where \( h \in \mathbb{R} \) is a parameter, determine if \( f_h \) is diagonalizable for some values of \( h \). For such values of \( h \), find a basis of \( \mathbb{R}^3 \) of eigenvectors of \( f_h \).

**Solution.** We consider the canonical basis \( C \) of \( \mathbb{R}^3 \), \( C = \{(1,0,0),(0,1,0),(0,0,1)\} \), and evaluate the matrix \( A_h \) associated to the given linear transformation \( f_h \) with respect to \( C \). For this purpose, we need to calculate:

\[
f_h(1,0,0) = (1, 1, -h) \\
f_h(0,1,0) = (0, 1, 0) \\
f_h(0,0,1) = (-h, -h, 1).
\]

Now we can write \( A_h = \begin{pmatrix} 1 & 0 & -h \\ 1 & 1 & -h \\ -h & 0 & 1 \end{pmatrix} \). The eigenvalues and eigenvectors of \( f_h \) correspond with the eigenvalues and eigenvectors of \( A_h \), and \( f_h \) is diagonalizable if and only if \( A_h \) is diagonalizable. We look for eigenvalues of \( A_h \) solving the characteristic equation \( \det(A_h - \lambda I) = 0 \):

\[
det \begin{pmatrix} 1 & 0 & -h \\ 1 & 1 & -h \\ -h & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \lambda & 0 & -h \\ 0 & 1 - \lambda & -h \\ -h & 0 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^3 - (1 - \lambda)h^2 = (1 - \lambda)[(1 - \lambda)^2 - h^2] = (1 - \lambda)(1 - \lambda + h)(1 - \lambda - h).
\]

The solutions of the characteristic equation are \( \lambda = 1 \), \( \lambda = 1 - h \), and \( \lambda = 1 + h \), thus for any \( h \neq 0 \), the found eigenvalues are distinct, each with algebraic multiplicity one, and thus the corresponding \( f_h \) is diagonalizable. Let’s find the eigenvectors of \( f_h \), for any \( h \in \mathbb{R} \setminus \{0\} \).

For \( \lambda = 1 \), we must find the nontrivial solutions of the homogeneous system \((A_h - \lambda I)v = 0\), where \( \lambda = 1 \). The augmented matrix of the system is

\[
\begin{pmatrix} 0 & 0 & -h \\ 1 & 0 & -h \\ -h & 0 & 0 \end{pmatrix}.
\]

All the eigenvectors corresponding to \( \lambda = 1 \) are \( \{t(0,1,0) : t \in \mathbb{R} \setminus \{0\} \} \), and the eigenspace corresponding to \( \lambda = 1 \) is \( U_1 = \text{span}\{(0,1,0)\} \).
For $\lambda = 1 - h$, we must find the nontrivial solutions of the homogeneous system $(A_h - \lambda I)v = 0$, where $\lambda = 1 - h$. The augmented matrix of the system is

$$
\begin{pmatrix}
  h & 0 & -h & 0 \\
  1 & h & -h & 0 \\
  -h & 0 & h & 0
\end{pmatrix}.
$$

All the eigenvectors corresponding to $\lambda = 1 - h$ are $\{t(1,1 - \frac{1}{h},1) : t \in \mathbb{R} \setminus \{0\}\}$, and the eigenspace corresponding to $\lambda = 1 - h$ is $U_{1-h} = \text{span}\{(1,1 - \frac{1}{h},1)\}$.

For $\lambda = 1 + h$, we must find the nontrivial solutions of the homogeneous system $(A_h - \lambda I)v = 0$, where $\lambda = 1 + h$. The augmented matrix of the system is

$$
\begin{pmatrix}
  -h & 0 & -h & 0 \\
  1 & -h & -h & 0 \\
  -h & 0 & -h & 0
\end{pmatrix}.
$$

All the eigenvectors corresponding to $\lambda = 1 + h$ are $\{t(1,1 + \frac{1}{h},-1) : t \in \mathbb{R} \setminus \{0\}\}$, and the eigenspace corresponding to $\lambda = 1 + h$ is $U_{1+h} = \text{span}\{(1,1 + \frac{1}{h},-1)\}$.

For any $h \neq 0$, we found a basis $B$ of $\mathbb{R}^3$ made of eigenvectors of $f_h$:

$$B = \{(0,1,0),(1,1 - \frac{1}{h},1),(1,1 + \frac{1}{h},-1)\}.$$

Let’s now analyse the case when $h = 0$, and determine if $A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is diagonalizable. We have already calculated the eigenvalues of $A_0$: we found that $\lambda = 1$ is the only eigenvalue with algebraic multiplicity three.

Let’s now evaluate the eigenvectors corresponding to $\lambda = 1$. We must find the nontrivial solutions of the homogeneous system $(A_0 - \lambda I)v = 0$, where $\lambda = 1$. The augmented matrix of the system is

$$
\begin{pmatrix}
  0 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}.
$$

All eigenvalues of $A_0$ are $\{t(0,1,0) + u(0,0,1) : t, u \in \mathbb{R} \text{ non both zero}\}$, and the eigenspace corresponding to $\lambda = 1$ is $U_1 = \text{span}\{(0,1,0),(0,0,1)\}$. Since the eigenvalue $\lambda = 1$ has algebraic multiplicity three, but the corresponding eigenspace has dimension two, we conclude that $A_0$, and thus $f_0$, is not diagonalizable. There is no basis of $\mathbb{R}^3$ formed by eigenvectors of $f_0$.

### 6.13) Find the linear transformation $l : \mathbb{R}^3 \to \mathbb{R}^3$ stretching every vector on the plane $x + 2y + 3z = 0$ twice and reversing the vector $(0, -1, 1)$.

#### Solution.

We choose two linearly independent vectors lying on the plane $x + 2y + 3z = 0$, for example $(2, -1, 0)$ and $(3, 0, -1)$ (both vectors have coordinates that satisfy the equation of the plane, and since they are not multiple of each other, they are linearly independent). The vector $(0, 1, 1)$ does not satisfy the equation of the plane, thus does not lie on the plane, and it is linearly independent with the chosen vectors. This implies that $B = \{(2, -1, 0), (3, 0, -1), (0, 1, -1)\}$ is a basis of $\mathbb{R}^3$, moreover $B$ is formed by eigenvectors and in particular:

$$
l(2, -1, 0) = 2(2, -1, 0) \quad l(3, 0, -1) = 2(3, 0, -1) \quad l(0, 1, 1) = - (0, 1, 1),
$$

thus the matrix $D = A(l, B, B)$ associated to the linear transformation with respect to the basis $B$ is diagonal.

We can write $D = A(l, B, B) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. In order to find the matrix $A = A(l, C, C)$ associated to $l$ with respect to the canonical basis $C$ of $\mathbb{R}^3$, $C = \{(1,0,0),(0,1,0),(0,0,1)\}$, we just recall that

$$D = P^{-1}AP$$

where $P = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ is the transition matrix from basis $B$ to basis $C$, and thus

$$A = PDP^{-1}.$$
In order to evaluate $P^{-1}$ we use a Gauss reduction:

$$
P = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} R_1 + 2R_2
$$

$$
\begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & -2 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} 3R_3 + R_2
$$

$$
\begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix} R_2 + 2R_3
$$

$$
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -6 & -6 \\ 3 & 6 & 6 \\ 1 & 2 & 3 \end{pmatrix} R_1/3
$$

and

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -3 & -3 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = P^{-1}.
$$

We are now ready to find the matrix $A = A(l, C, C)$ associated to $l$ with respect to the canonical basis $C$ of $\mathbb{R}^3$:

$$
A = PDP^{-1} = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & -3 & -3 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 8 & 8 \\ -3 & -6 & -7 \end{pmatrix}.
$$

$l : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as follows:

$$
l(x, y, z) = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 8 & 8 \\ -3 & -6 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (2x, 3x + 8y + 9z, -3x - 6y - 7z).
$$

6.15) Find the linear transformation $l : \mathbb{R}^2 \to \mathbb{R}^2$ knowing that $(1, 2)$ and $(1, 3)$ are two of its eigenvectors and $l(1, 0) = (4, 6)$

**Solution.** We must determine the two eigenvalues $\lambda_1, \lambda_2$ so that $l(1, 2) = \lambda_1(1, 2)$ and $l(1, 3) = \lambda_2(1, 3)$. Then we proceed in finding the matrix $A$ associated to $l$ with respect to the canonical basis $C = \{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$, following the same method used in Solution of Exercise 13. In order to find $\lambda_1$ and $\lambda_2$, we can use the only information given about $l$, $l(1, 0) = (4, 6)$, and the linearity of $l$. Let’s first write the vector $(1, 0)$ as a linear combination of $(1, 2)$, and $(1, 3)$, i.e., we look for numbers $a, b \in \mathbb{R}$ so that

$$(1, 0) = a(1, 2) + b(1, 3).$$

This is equivalent to solving the system:

$$
a + b = 1 \\
2a + 3b = 0.
$$

After simple calculation, we find $a = 3$, and $b = -2$ as the only possible solution of the system, thus $(1, 0) = 3(1, 2) - 2(1, 3)$. We now use the linearity of $l$, the fact that $(1, 2)$ and $(1, 3)$ are eigenvectors associated to $\lambda_1$ and $\lambda_2$, and write:

$$(4, 6) = l(1, 0) = l(3(1, 2) - 2(1, 3)) = 3l(1, 2) - 2l(1, 3) = 3\lambda_1(1, 2) - 2\lambda_2(1, 3) = (3\lambda_1 - 2\lambda_2, 6\lambda_1 - 6\lambda_2).
$$

This allows us to find $\lambda_1$ and $\lambda_2$, solving the system:

$$
3\lambda_1 - 2\lambda_2 = 4 \\
6\lambda_1 - 6\lambda_2 = 6.
$$

After simple calculation, we find $\lambda_1 = 2$ and $\lambda_2 = 1$.

We now know that $l(1, 2) = 2(1, 2)$, $l(1, 3) = 3(1, 3)$. Considering the basis $B = \{(1, 2), (1, 3)\}$ of $\mathbb{R}^2$, the matrix associate to $l$ with respect to $B$ is diagonal, $D = A(l, B, B) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. In order to find the matrix $A = A(l, C, C)$ associated to $l$ with respect to the canonical basis $C$ of $\mathbb{R}^2$, $C = \{(1, 0), (0, 1)\}$, we recall that

$$
D = P^{-1}AP
$$

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where \( P = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \) is the transition matrix from basis \( B \) to basis \( C \), and thus
\[
A = PDP^{-1}.
\]

We immediately evaluate \( P^{-1} = \frac{1}{\det P} \text{adj} P = \begin{pmatrix} \frac{1}{3} & -\frac{1}{2} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \). We are now ready to find the matrix \( A \):
\[
A = A(I, C, C) = PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -12 & 8 \end{pmatrix}.
\]

\( l : \mathbb{R}^2 \to \mathbb{R}^2 \) is defined as follows:
\[
l(x, y) = \begin{pmatrix} -2 \\ -12 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (-2x + 2y, -12x + 8y).
\]

6.17) Given \( A = \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \), use the eigenvalue method
   a) to derive an explicit formula for \( A^n \),
   b) to find a general solution of the system of differential equations
   \[
   \frac{dx}{dt} = 2x - 3y,
   \]
   \[
   \frac{dx}{dt} = 4x - 5y.
   \]

Solution. a) We start by finding the eigenvalues of \( A \), solving the characteristic equation \( \det(A - \lambda I) = 0 \), we get
\[
\det \begin{pmatrix} 2-\lambda & -3 \\ -5 & -\lambda \end{pmatrix} = (2 - \lambda)(-5 - \lambda) + 12 = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2).
\]

\( A \) has distinct eigenvalues \( \lambda_1 = -1 \), and \( \lambda_2 = -2 \). We find corresponding eigenvectors. For \( \lambda_1 = -1 \), we solve the homogeneous system with associated matrix \( A - \lambda_1 I \):
\[
\begin{pmatrix} 3 & -3 & 0 \\ 4 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix}.
\]

The eigenspace associated to \( \lambda_1 = -1 \) is \( U_{-1} = \text{span}\{(1, 1)\} \). For \( \lambda_2 = -2 \), we solve the homogeneous system with associated matrix \( A - \lambda_2 I \):
\[
\begin{pmatrix} 4 & -3 & 0 \\ 4 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 4 & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix}.
\]

The eigenspace associated to \( \lambda_2 = -2 \) is \( U_{-2} = \text{span}\{(3, 4)\} \). Hence, if \( P = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \), we have \( P^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} \), and
\[
P^{-1}AP = D = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.
\]

Thus
\[
A^n = (PDP^{-1})^n = PD^nP^{-1} = PDP^{-1}PDP^{-1} \ldots PDP^{-1} = PD^nP^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & (-2)^n \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} = (-1)^n \begin{pmatrix} 1 & 3 \cdot 2^n \\ 1 & 4 \cdot 2^n \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} = (-1)^n \begin{pmatrix} 4 - 3 \cdot 2^n & -3 + 3 \cdot 2^n \\ 4 - 4 \cdot 2^n & -3 + 4 \cdot 2^n \end{pmatrix} = (-1)^n \begin{pmatrix} 4 & -3 \\ 4 & -3 \end{pmatrix} + (-2)^n \begin{pmatrix} -3 & 3 \\ -4 & 4 \end{pmatrix}.
\]

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b) Consider a linear system of differential equations $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{x}'(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $x_1(t), x_2(t), \ldots, x_n(t)$ are unknown functions, $\mathbf{A}$ is a real $n \times n$ matrix, and $\mathbf{x}'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t))^T$. We recall that, if $\mathbf{A}$ has $n$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are eigenvectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$, then a general solution $\mathbf{x}(t)$ of the system is

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}, \quad c_1, \ldots, c_n \in \mathbb{R}.$$

We now find a general solution of the system of differential equations

$$\begin{align*}
\frac{dx}{dt} &= 2x - 3y \\
\frac{dy}{dt} &= 4x - 5y.
\end{align*}$$

In a) we found $\lambda_1 = -1$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\lambda_2 = -2$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Hence, the general solution of the given system is:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{-2t} = \begin{pmatrix} c_1 e^{-t} + 3c_2 e^{-2t} \\ c_1 e^{-t} + 4c_2 e^{-2t} \end{pmatrix}.$$

6.19) Solve the system of recurrence relations

$$\begin{align*}
x_{n+1} &= 2x_n - y_n \\
y_{n+1} &= -x_n + 2y_n
\end{align*}$$

given that $x_0 = 0$ and $y_0 = -1$.

Solution. Let’s consider $\mathbf{X} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$. Then, the given system can be written in matrix form as

$$\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n,$$

where

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{X}_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

We can prove, with an easy induction, that:

$$\begin{align*}
\mathbf{X}_1 &= \mathbf{A}\mathbf{X}_0 \\
\mathbf{X}_2 &= \mathbf{A}\mathbf{X}_1 = \mathbf{A}\mathbf{A}\mathbf{X}_0 = \mathbf{A}^2\mathbf{X}_0 \\
&\vdots \\
\mathbf{X}_n &= \mathbf{A}^n\mathbf{X}_0
\end{align*}$$

We thus need to evaluate $\mathbf{A}^n$, and we will do so using the eigenvalues and eigenvectors of $\mathbf{A}$ as in Solution of Exercise 17.

We start by finding the eigenvalues of $\mathbf{A}$, solving the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, we get

$$\begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} = (2 - \lambda)^2 - 1 = (2 - \lambda - 1)(2 - \lambda + 1) = (\lambda - 1)(\lambda - 3).$$

$\mathbf{A}$ has distinct eigenvalues $\lambda_1 = 1$, and $\lambda_2 = 3$. We find corresponding eigenvectors. For $\lambda_1 = 1$, we solve the homogeneous system with associated matrix $\mathbf{A} - \lambda_1\mathbf{I}$:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{R}_1 + \mathbf{R}_2 \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

The eigenspace associated to $\lambda_1 = 1$ is $U_1 = \text{span}\{(1, 1)\}$, thus $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda_2 = 3$, we solve the homogeneous system with associated matrix $\mathbf{A} - \lambda_2\mathbf{I}$:

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \mathbf{R}_2 - \mathbf{R}_1 \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$
The eigenspace associated to $\lambda_2 = 3$ is $U_3 = \text{span}\{(1, -1)\}$, thus $\vec{v}_2 = \left( \begin{array}{c} 1 \\ -1 \end{array} \right)$. Hence, if $P = \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right)$, we have $P^{-1} = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right)$, and

$$P^{-1}AP = D = \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right).$$

Thus

$$A^n = PD^n P^{-1} =$$

$$= \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right)^n \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) =$$

$$= \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 3^n \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) =$$

$$= \frac{1}{2} \left( \begin{array}{cc} 1 & 3^n \\ 1 & -3^n \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) =$$

$$= \frac{1}{2} \left( \begin{array}{cc} 1 + 3^n & 1 - 3^n \\ 1 - 3^n & 1 + 3^n \end{array} \right).$$

We now conclude evaluating the solution of the given system of recurrence relations:

$$\left( \begin{array}{c} x_n \\ y_n \end{array} \right) = X_n = A^n X_0 = \frac{1}{2} \left( \begin{array}{cc} 1 + 3^n & 1 - 3^n \\ 1 - 3^n & 1 + 3^n \end{array} \right) \left( \begin{array}{c} 0 \\ -1 \end{array} \right) = \left( \begin{array}{c} \frac{-1 + 3^n}{2} \\ \frac{-1 - 3^n}{2} \end{array} \right).$$

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Chapter 7
Euclidean spaces

7.1) Given the vectors \( \mathbf{v} = (1, 5), \mathbf{u} = (3, 4) \) in \( \mathbb{R}^2 \), with the standard inner product, find \( \langle \mathbf{u}, \mathbf{v} \rangle, \|\mathbf{u}\|, \|\mathbf{v}\| \).

7.2) Given the vectors \( \mathbf{v} = (1, 5, 2), \mathbf{u} = (3, 4, 0) \) in \( \mathbb{R}^3 \), with the standard inner product, find \( \langle \mathbf{u}, \mathbf{v} \rangle, \|\mathbf{u}\|, \|\mathbf{v}\| \).

7.3) Verify that, on \( \mathbb{R}^2, \langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 \), where \( \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \), is a well defined inner product.

7.4) In \( \mathcal{P}^2 \) with the standard inner product, i.e. \( \langle f, g \rangle = \int_0^1 f(t)g(t) \, dt \), consider

\[
\begin{align*}
f(t) &= t - 3, & g(t) &= -\frac{15}{7} t + 1, & h(t) &= 3t - 2.
\end{align*}
\]

Verify that \( f \) is orthogonal to \( g \) and \( h \) has norm 1.

7.5) In \( C[0,1] \) with the standard inner product, i.e. \( \langle f, g \rangle = \int_0^1 f(t)g(t) \, dt \), consider the functions

\[
\begin{align*}
f(t) &= t + 2, & g(t) &= 3t - 2, & h(t) &= e^t.
\end{align*}
\]

Find \( \langle f, g \rangle, \langle f, h \rangle, ||f||, ||h|| \).

7.6) In \( \mathbb{R}^3 \) verify that the vectors \( \mathbf{x} = (2, 3, 5) \) and \( \mathbf{y} = (1, -4, 3) \) are not orthogonal but form an acute angle.

7.7) In \( \mathbb{R}^3 \) are given \( \mathbf{u} = (1, 1, 1), \mathbf{v} = (1, 2, -3), \mathbf{w} = (1, -4, 3) \), verify that \( \mathbf{u} \) is orthogonal to \( \mathbf{v} \), \( \mathbf{u} \) is orthogonal to \( \mathbf{w} \), but \( \mathbf{v} \) is not orthogonal to \( \mathbf{w} \).

7.8) Find the angle \( \theta \) between the vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^4, \mathbf{u} = (1, 3, -5, 4), \mathbf{v} = (2, -3, 4, 1) \).

7.9) Find the angle \( \theta \) between the vectors \( \mathbf{A}, \mathbf{B} \in \mathbb{M}^{2,3} \), if \( \mathbf{A} = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \), \( \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \).

7.10) Find \( k \in \mathbb{R} \) such that \( \mathbf{u} = (1, 2, k, 3) \) and \( \mathbf{v} = (3, k, 7, -5) \) are orthogonal in \( \mathbb{R}^4 \).

7.11) Find a nonzero vector that is orthogonal to \( \mathbf{u} = (1, 2, 1) \) and \( \mathbf{v} = (2, 5, 4) \) in \( \mathbb{R}^3 \).

7.12) Find \( k \in \mathbb{R} \) such that \( \mathbf{A} = \begin{pmatrix} 1 & 3 & k \\ 1 & 1 & 1 \end{pmatrix} \) and \( \mathbf{B} = \begin{pmatrix} k & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix} \) are orthogonal in \( \mathbb{M}^{2,3} \).

7.13) Verify that \( \cos t, \sin t \) are orthogonal vectors in \( C[0, 2\pi] \), with respect to the standard inner product.

7.14) In \( C[0, 2\pi] \), prove that the set \( S = \{1, \sin t, \cos t, \sin 2t, \cos 2t, \ldots, \sin kt, \cos kt, \ldots\} \) is an orthogonal set, normalize each vector of \( S \) to create an orthonormal set.

7.15) Given, in \( \mathbb{R}^3, W = \text{span}\{(1, 2, 3, -1, 2), (2, 4, 7, 2, -1)\} \), find a basis of the orthogonal complement \( W^\perp \).

7.16) In \( \mathbb{R}^4 \) is given the space \( W = \text{span}\{(1, 2, 3, 1)\} \), find an orthogonal basis of the orthogonal complement \( W^\perp \).

7.17) For the subspace \( \mathbf{u}^\perp \) of \( \mathbb{R}^3 \), where \( \mathbf{u} = (1, 3, -4) \), find a basis, an orthogonal basis, an orthonormal basis.

7.18) Find the Fourier coefficient \( c = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{||\mathbf{w}||^2} \) and the projection of \( \mathbf{v} = (1, -2, 3, -4) \) along \( \mathbf{w} = (1, 2, 1, 2) \) with respect to the standard inner product in \( \mathbb{R}^4 \).

7.19) In \( \mathbb{R}^3 \) verify that \( B = \{\mathbf{u}_1 = (1, 2, 1), \mathbf{u}_2 = (2, 1, -4), \mathbf{u}_3 = (3, -2, 1)\} \) is an orthogonal basis, find \( [\mathbf{v}]_B \), the coordinates of \( \mathbf{v} = (7, 1, 9) \) with respect to \( B \).

7.20) In \( \mathbb{R}^4 \) is given the set \( S = \{(1, 1, 0, -1), (1, 2, 1, 3), (1, 1, -9, 2), (16, -13, 1, 3)\} \). Show that \( S \) is orthogonal, and it forms a basis of \( \mathbb{R}^4 \). Find the coordinates of \( \mathbf{v} = (a, b, c, d) \) with respect to \( S \).

7.21) In \( \mathbb{R}^4 \) consider \( U = \text{span}\{(1, 1, 1, 1), (1, 1, 2, 4), (1, 2, -4, -3)\} \). Use the Gram-Schmidt algorithm to find an orthogonal basis of \( U \), then find an orthonormal basis of \( U \).
7.22) Consider the space $P^2(t)$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt$.
   a) Find $\langle f, g \rangle$ where $f(t) = t + 2$, $g(t) = t^2 - 3t + 4$.
   b) Apply Gram-Schmidt algorithm to the set $\{1, t, t^2\}$ to obtain an orthogonal set \{f_0, f_1, f_2\} of polynomials with integer coefficients.

7.23) Prove that every symmetric $2 \times 2$ matrix is diagonalizable. Given $l : \mathbb{R}^2 \to \mathbb{R}^2$, $l(x, y) = (2x + 2y, 2x + 5y)$, show that $l$ is diagonalizable and find a basis of $\mathbb{R}^2$ made of orthonormal eigenvectors of $l$.

7.24) Given the symmetric matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, find an orthonormal real matrix $P$ such that $P^tAP$ is diagonal.

( Remember that $P$ is constructed with orthonormal eigenvectors of $A$.)

7.25) Verify that the linear transformation $l : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $l(x, y, z) = (x + 3y + 4z, 3x + y, 4x + z)$ is symmetric, prove that it is diagonalizable and there exists a basis of $\mathbb{R}^3$ made of three orthogonal eigenvectors of $l$.

7.26) Identify the curve with equation $x^2 - 4xy + 4y^2 + 5y - 9 = 0$.

7.27) Identify the curve with equation $12x^2 - 12xy + 7y^2 + 60x - 38y + 31 = 0$. 
Solutions of odd number problems - Euclidean spaces

7.1) Given the vectors \( \mathbf{v} = (1, 5), \mathbf{u} = (3, 4) \) in \( \mathbb{R}^2 \), with the standard inner product, find \( \langle \mathbf{u}, \mathbf{v} \rangle \), \( ||\mathbf{u}|| \), \( ||\mathbf{v}|| \).

**Solution.** On \( \mathbb{R}^n \), given \( \mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \), we define the standard inner product \( \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} x_i y_i \).

Moreover, we define the norm of \( \mathbf{x} \) as the nonnegative number \( ||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \). Thus:

\[
\langle \mathbf{u}, \mathbf{v} \rangle = \langle (1, 5), (3, 4) \rangle = 1 \cdot 3 + 5 \cdot 4 = 3 + 20 = 23,
\]

\[
||\mathbf{u}|| = \sqrt{1^2 + 5^2} = \sqrt{26},
\]

\[
||\mathbf{v}|| = \sqrt{3^2 + 4^2} = 5.
\]

7.3) Verify that, on \( \mathbb{R}^2 \), \( \langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 \), where \( \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \), is a well defined inner product.

**Solution.** By definition, given a linear space \( L \) on \( \mathbb{R} \), we call inner product on \( L \) a function \( \langle \cdot, \cdot \rangle : L \times L \to \mathbb{R} \), \( \langle \cdot, \cdot \rangle : (\mathbf{x}, \mathbf{y}) \to \langle \mathbf{x}, \mathbf{y} \rangle \) satisfying:

i) \( \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle \) for every \( \mathbf{x}, \mathbf{y} \in L \),

ii) \( \langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle \) for every \( \mathbf{x}, \mathbf{y}, \mathbf{z} \in L \),

iii) \( \langle c\mathbf{x}, \mathbf{y} \rangle = c \langle \mathbf{x}, \mathbf{y} \rangle \) for every \( \mathbf{x}, \mathbf{y} \in L \), and every \( c \in \mathbb{R} \),

iv) \( \langle \mathbf{x}, \mathbf{x} \rangle \geq 0 \) for every \( \mathbf{x} \in L \), and \( \langle \mathbf{x}, \mathbf{x} \rangle = 0 \) if and only if \( \mathbf{x} = 0_L \).

The given inner product on \( \mathbb{R}^2 \), can be defined with the use of the matrix product as follows:

\[
\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 = (x_1 \quad x_2)^T \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1 \quad x_2) A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
\]

where \( A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{M}^{2 \times 2} \). Due to the properties of matrix product and the fact that \( A \) is symmetric (\( A = A^T \)), we can easily verify that all properties i)–iv) are satisfied. Indeed:

i) \( \langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 = 2y_1x_1 + y_1x_2 + y_2x_1 + y_2x_2 = \langle \mathbf{y}, \mathbf{x} \rangle \),

ii) \( \langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = 2x_1(y_1 + z_1) + x_1(y_2 + z_2) + x_2(y_1 + z_1) + x_2(y_2 + z_2) = 2(x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2) + (2x_1z_1 + x_1z_2 + x_2z_1 + x_2z_2) = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle \),

iii) \( \langle c\mathbf{x}, \mathbf{y} \rangle = c(2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2) = 2cx_1y_1 + cx_1y_2 + cx_2y_1 + cx_2y_2 = c \langle \mathbf{x}, \mathbf{y} \rangle \),

iv) \( \langle \mathbf{x}, \mathbf{x} \rangle = 2x_1^2 + x_1x_2 + x_2x_1 + x_2^2 = 2x_1^2 + 2x_1x_2 + x_2^2 = x_1^2 + (x_1 + x_2)^2 \geq 0 \)

and \( \langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + (x_1 + x_2)^2 = 0 \) if and only if \( x_1 = x_2 = 0 \), i.e. \( \mathbf{x} = (0, 0) \).

7.5) In \( \mathbb{C}[0, 1] \) with the standard inner product, i.e. \( \langle f, g \rangle = \int_0^1 f(t)g(t) \, dt \), consider the functions

\[
f(t) = t + 2, \quad g(t) = 3t - 2, \quad h(t) = e^t.
\]

Find \( \langle f, g \rangle, \langle f, h \rangle, \langle f, f \rangle, \langle h, h \rangle \).

**Solution.** We evaluate:

\[
\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt = \int_0^1 (t + 2)(3t - 2) \, dt = \left[ t^3 + 2t^2 - 4t \right]_0^1 = -1.
\]

\[
\langle f, h \rangle = \int_0^1 f(t)h(t) \, dt = \int_0^1 (t + 2)e^t \, dt = \left[ e^t(t + 1) \right]_0^1 = 2e - 1.
\]

\[
\langle f, f \rangle = \int_0^1 f(t)^2 \, dt = \int_0^1 (t + 2)^2 \, dt = \left[ \frac{t^3}{3} + 2t^2 + 4t \right]_0^1 = \frac{19}{3}.
\]

\[
\langle h, h \rangle = \int_0^1 h(t)^2 \, dt = \int_0^1 (e^t)^2 \, dt = \int_0^1 e^{2t} \, dt = \left[ \frac{e^{2t}}{2} \right]_0^1 = \frac{1}{2}(e^2 - 1).
\]
7.7) In \( \mathbb{R}^3 \) are given the vectors \( \mathbf{u} = (1, 1, 1), \mathbf{v} = (1, 2, -3), \mathbf{w} = (1, -4, 3) \). Verify that \( \mathbf{u} \) is orthogonal to \( \mathbf{v} \), \( \mathbf{u} \) is orthogonal to \( \mathbf{w} \), but \( \mathbf{v} \) is not orthogonal to \( \mathbf{w} \).

**Solution.** Since no particular inner product is given, it is understood that we consider \( \mathbb{R}^3 \) with the standard inner product. By definition, two vectors \( \mathbf{x}, \mathbf{y} \) in an Euclidean space \( V \) are orthogonal if \( \langle \mathbf{x}, \mathbf{y} \rangle = 0 \), thus we evaluate:

\[
\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = (1, 1, 1) \cdot (1, 2, -3) = 1 + 2 - 3 = 0, \text{ i.e. } \mathbf{u} \perp \mathbf{v},
\]

\[
\langle \mathbf{u}, \mathbf{w} \rangle = \mathbf{u} \cdot \mathbf{w} = (1, 1, 1) \cdot (1, -4, 3) = 1 - 4 + 3 = 0, \text{ i.e. } \mathbf{u} \perp \mathbf{w},
\]

\[
\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = (1, 2, -3) \cdot (1, -4, 3) = 1 - 8 + 9 = 2, \text{ i.e. } \mathbf{v} \text{ is not orthogonal to } \mathbf{w}.
\]

7.9) Find the angle \( \theta \) between the vectors \( \mathbf{A}, \mathbf{B} \in M^{2,3} \), if \( \mathbf{A} = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \).

**Solution.** We recall that in \( M^{m,n} \), the standard inner product between two matrices \( \mathbf{A}, \mathbf{B} \) is defined as \( \langle \mathbf{A}, \mathbf{B} \rangle = \text{tr}(\mathbf{B}^T \mathbf{A}) \), where "\( \text{tr} \)" stands for "trace", i.e. the sum of the diagonal entries of a matrix. Moreover, in a real Euclidean space we define the angle between two nonzero vectors \( \mathbf{u}, \mathbf{v} \) as the angle \( 0 \leq \theta \leq \pi \) such that \( \cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}|| \cdot ||\mathbf{v}||} \). Thus we need to evaluate

\[
\langle \mathbf{A}, \mathbf{B} \rangle = \text{tr}(\mathbf{B}^T \mathbf{A}) = \text{tr} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} = \text{tr} \begin{pmatrix} 9 + 24 & \cdots & \cdots \\ \cdots & 16 + 25 & \cdots \\ \cdots & \cdots & 21 + 24 \end{pmatrix} = 119,
\]

\[
\|\mathbf{A}\|^2 = \langle \mathbf{A}, \mathbf{A} \rangle = \text{tr}(\mathbf{A}^T \mathbf{A}) = \text{tr} \begin{pmatrix} 9 & 6 \\ 8 & 5 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} = \text{tr} \begin{pmatrix} 81 + 36 & \cdots & \cdots \\ \cdots & 64 + 25 & \cdots \\ \cdots & \cdots & 49 + 16 \end{pmatrix} = 271,
\]

\[
\|\mathbf{B}\|^2 = \langle \mathbf{B}, \mathbf{B} \rangle = \text{tr}(\mathbf{B}^T \mathbf{B}) = \text{tr} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \text{tr} \begin{pmatrix} 1 + 16 & \cdots & \cdots \\ \cdots & 4 + 25 & \cdots \\ \cdots & \cdots & 9 + 36 \end{pmatrix} = 91,
\]

and finally

\[
\cos \theta = \frac{\langle \mathbf{A}, \mathbf{B} \rangle}{\|\mathbf{A}\| \cdot \|\mathbf{B}\|} = \frac{119}{\sqrt{271} \cdot \sqrt{91}} \simeq 0,75777,
\]

from which we get: \( \theta \simeq \arccos(0,75777) \).

7.11) Find a nonzero vector that is orthogonal to \( \mathbf{u} = (1, 2, 1) \) and \( \mathbf{v} = (2, 5, 4) \) in \( \mathbb{R}^3 \).

**Solution.** We are looking for a vector \( \mathbf{x} = (x, y, z) \in \mathbb{R}^3 \) such that \( \mathbf{x} \cdot \mathbf{u} = 0 \) and \( \mathbf{x} \cdot \mathbf{v} = 0 \), thus the coordinates of \( \mathbf{x} \) are a nontrivial solutions of the homogeneous system:

\[
\begin{align*}
2x + 2y + z &= 0 \\
2x + 5y + 4z &= 0
\end{align*}
\]

We may easily guess a solution: by setting \( z = 1 \), we get \( y = -2 \) and subsequently \( x = 3 \). The vector \( \mathbf{x} = (3, -2, 1) \), or any multiple of \( \mathbf{x} \) is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).

7.13) Verify that \( \cos t, \sin t \) are orthogonal vectors in \( C[0,2\pi] \), with respect to the standard inner product.

**Solution.** In \( C[0,2\pi] \) the standard inner product is \( \langle f, g \rangle = \int_0^{2\pi} f(t)g(t) \, dt \), thus we just verify that:

\[
\int_0^{2\pi} \cos t \sin t \, dt = \frac{1}{2} \left( -\sin^2 t \right) \bigg|_0^{2\pi} = \frac{1}{2} \left( 0 - 0 \right) = 0 - 0 = 0,
\]

and this implies that \( \cos t, \sin t \) are orthogonal vectors in \( C[0,2\pi] \).

7.15) Given, in \( \mathbb{R}^5 \), \( W = \text{span}\{ (1,2,3,-1,2), (2,4,7,2,-1) \} \), find a basis of the orthogonal complement \( W^\perp \).

**Solution.** Let \( W \) be a subset of an Euclidean space \( E \), then \( W^\perp \) is defined as follows:

\[
W^\perp = \{ \mathbf{v} \in E : \mathbf{v} \perp \mathbf{u}, \text{ for every } \mathbf{u} \in W \}.
\]
Moreover, we recall that \( W^\perp \) is a subspace of \( E \). In case \( W \) is a subspace of \( E \), then \( E = W \oplus W^\perp \), we say that \( E \) is the direct sum of \( W \) and \( W^\perp \), i.e. every \( v \in E \) can be written in one and only one way as \( v = w + w^\perp \), where \( w \in W \) and \( w^\perp \in W^\perp \).

We need to find a basis of \( W^\perp = \left[ \text{span}\{(1,2,3,-1,2),(2,4,7,2,-1)\} \right]^\perp \). We are looking for all possible vectors \( x = (x,y,z,t,w) \in \mathbb{R}^5 \), so that:

\[
\begin{align*}
(1,2,3,-1,2) \cdot (x,y,z,t,w) &= 0 \\
(2,4,7,2,-1) \cdot (x,y,z,t,w) &= 0.
\end{align*}
\]

We are looking for a basis of the solution set of the homogeneous system:

\[
\begin{align*}
x + 2y + 3z - t + 2w &= 0 \\
2x + 4y + 7z + 2t - w &= 0 \\
R_2 - 2R_1 &= x + 2y + 3z - t + 2w = 0 \\
z + 4t - 5w &= 0.
\end{align*}
\]

Choosing \( y = \alpha \), \( t = \beta \), \( w = \gamma \), as parameters, \( \alpha, \beta, \gamma \in \mathbb{R} \), from the last equation we get \( z = -4\beta + 5\gamma \), that substituted into the first equation gives \( x = -2\alpha + 13\beta - 17\gamma \). Thus, all solutions of the previous homogeneous system can be written as

\[
x = (x,y,z,t,w) = (-2\alpha + 13\beta - 17\gamma, \alpha, -4\beta + 5\gamma, \beta, \gamma) = \\
= \alpha(-2,1,0,0,0) + \beta(13,0,-4,1,0) + \gamma(-17,0,5,0,1).
\]

Thus

\[
W^\perp = \text{span}\{(-2,1,0,0,0),(13,0,-4,1,0),(-17,0,5,0,1)\}.
\]

### 7.17
For the subspace \( u^\perp \) of \( \mathbb{R}^3 \), where \( u = (1,3,-4) \), find a basis, an orthogonal basis, an orthonormal basis.

**Solution.** We recall that, given an Euclidean space \( V \), and a non-zero vector \( u \in V \), the set \( u^\perp = \{ v \in V : u \perp v \} \) is a subspace of \( V \). In our case, we may denote \( U = \text{span}\{u\} \), then, since \( \dim U = 1 \), and \( \mathbb{R}^3 = U \oplus u^\perp \), we deduce that \( u^\perp \) has dimension two. To find a basis of \( u^\perp \), we need two linearly independent vectors of \( \mathbb{R}^3 \), \( v_1 \) and \( v_2 \), that are both perpendicular to the given \( u \). We may choose the first \( v_1 = (x_1,y_1,z_1) \), just taking care that the condition

\[
u \cdot v_1 = 0,
\]

must be satisfied. Thus, for example, we may choose \( v_1 = (1,1,1) \). We now look for \( v_2 \), in a way that \( \{v_1,v_2\} \) will form an orthogonal basis of \( u^\perp \), so we look for \( v_2 = (x_2,y_2,z_2) \), satisfying both conditions

\[
u \cdot v_2 = (1,3,-4) \cdot (x_2,y_2,z_2) = 0,
\]

and

\[
v_1 \cdot v_2 = (1,1,1) \cdot (x_2,y_2,z_2) = 0,
\]

that is, satisfying the system:

\[
\begin{align*}
x_2 + 3y_2 - 4z_2 &= 0 \\
x_2 + y_2 + z_2 &= 0 \\
R_1 - R_2 &= x_2 + 3y_2 - 4z_2 = 0 \\
2y_2 - 5z_2 &= 0.
\end{align*}
\]

We may easily guess a possible solution \( v_2 = (-7,5,2) \). The set \( B = \{v_1,v_2\} = \{(1,1,1),(-7,5,2)\} \) is a basis of \( u^\perp \) made of orthogonal vectors, thus, it is an orthogonal basis of \( u^\perp \). To find an orthonormal basis of \( u^\perp \), it is sufficient to normalise \( v_1 \) and \( v_2 \).

\[
\tilde{B} = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|} \right\} = \left\{ \left(\frac{1,1,1}{\sqrt{3}}, \frac{-7,5,2}{\sqrt{78}} \right) \right\}
\]

is an orthonormal basis of \( u^\perp \).

### 7.19
In \( \mathbb{R}^3 \), verify that \( B = \{u_1 = (1,2,1), u_2 = (2,1,-4), u_3 = (3,-2,1)\} \) is an orthogonal basis, find \( [v]_B \), the coordinates of \( v = (7,1,9) \) with respect to \( B \).

**Solution.** Let’s recall an important Theorem: Given an Euclidean space \( V \) and \( B = \{v_1,\ldots,v_n\} \) an orthogonal basis of \( V \), then for every vector \( x \in V \),

\[
x = \sum_{i=1}^{n} c_i v_i \quad \text{where} \quad c_i = \frac{\langle x, v_i \rangle}{\langle v_i, v_i \rangle}.
\]
The coefficients $c_i = \frac{\langle x, v_i \rangle}{\langle v_i, v_i \rangle}$ are called Fourier coefficients of $x$ with respect to $B$, and the coordinates of $x$ with respect to $B$ are

$$[x]_B = (c_1, \ldots, c_n).$$

Let's verify that the given set $B = \{ u_1 = (1, 2, 1), u_2 = (2, 1, -4), u_3 = (3, -2, 1) \}$ is an orthogonal basis of $\mathbb{R}^3$. Since orthogonal vectors are linearly independent, it is enough to verify the given three vectors are orthogonal. Indeed:

$$u_1 \cdot u_2 = (1, 2, 1) \cdot (2, 1, -4) = 2 + 2 - 4 = 0,$$

$$u_1 \cdot u_3 = (1, 2, 1) \cdot (3, -2, 1) = 3 - 4 + 1 = 0,$$

$$u_3 \cdot u_2 = (3, -2, 1) \cdot (2, 1, -4) = 6 - 2 - 4 = 0.$$

Now we may evaluate the Fourier coefficients of $v = (7, 1, 9)$ with respect to $B$.

$$c_1 = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{(7, 1, 9) \cdot (1, 2, 1)}{(1, 2, 1) \cdot (1, 2, 1)} = \frac{7 + 2 + 9}{1 + 4 + 1} = 3,$$

$$c_2 = \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} = \frac{(7, 1, 9) \cdot (2, 1, -4)}{(2, 1, -4) \cdot (2, 1, -4)} = \frac{14 + 1 - 36}{4 + 1 + 16} = -1,$$

$$c_3 = \frac{\langle v, u_3 \rangle}{\langle u_3, u_3 \rangle} = \frac{(7, 1, 9) \cdot (3, -2, 1)}{(3, -2, 1) \cdot (3, -2, 1)} = \frac{21 - 2 + 9}{9 + 4 + 1} = 2.$$

We conclude that the coordinates of $v = (7, 1, 9)$ with respect to $B$ are:

$$[v]_B = (c_1, c_2, c_3) = (3, -1, 2).$$

### 7.21

In $\mathbb{R}^4$ consider $U = \text{span}\{(1, 1, 1, 1), (1, 1, 2, 4), (1, 2, -4, -3)\}$. Use the Gram-Schmidt algorithm to find an orthogonal basis of $U$, then find an orthonormal basis of $U$.

**Solution.** Given any basis $B = \{v_1, \ldots, v_n\}$ of a finite dimensional Euclidean space $V$, we may create a new orthogonal basis $S = \{y_1, \ldots, y_n\}$ of $V$ using the following process, the **Gram-Schmidt orthogonalization process**:

$$y_1 = v_1$$

$$y_2 = v_2 - \frac{\langle v_2, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1$$

$$y_3 = v_3 - \frac{\langle v_3, y_2 \rangle}{\langle y_2, y_2 \rangle} y_2 - \frac{\langle v_3, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1$$

$$\ldots$$

$$y_n = v_n - \frac{\langle v_n, y_{n-1} \rangle}{\langle y_{n-1}, y_{n-1} \rangle} y_{n-1} - \cdots - \frac{\langle v_n, y_2 \rangle}{\langle y_2, y_2 \rangle} y_2 - \frac{\langle v_n, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1.$$

In our case, $V$ is the given subspace of $\mathbb{R}^4$, $V = U = \text{span}\{(1, 1, 1, 1), (1, 1, 2, 4), (1, 2, -4, -3)\}$, and $B = \{(1, 1, 1, 1), (1, 1, 2, 4), (1, 2, -4, -3)\}$. The process can be used starting from any set of vectors, nevertheless, for the sake of exercise, we may easily verify that the given vectors $v_1 = (1, 1, 1, 1), v_2 = (1, 1, 2, 4), v_3 = (1, 2, -4, -3)$ are linearly independent, and thus form a basis of $V$. Indeed, using Gauss reduction and the symbol $\sim$ to denote the coincidence of the corresponding spans, we have:

$$\begin{align*}
(1, 1, 1, 1) & \sim (1, 1, 1, 1) \\
(1, 1, 2, 4) & \sim (0, 0, 1, 34) \\
(1, 2, -4, -3) & \sim (0, 1, -5, -4)
\end{align*}$$

Since the last three vectors are obviously linearly independent, it follows that the given ones are also linearly independent. We now proceed with the orthogonalization process:
7.23) Prove that every symmetric 2 \times 2 matrix is diagonalizable. Given l : \mathbb{R}^2 \to \mathbb{R}^2, l(x, y) = (2x + 2y, 2x + 5y), show that l is diagonalizable and find a basis of IR^2 made of orthonormal eigenvectors of l.

Solution. Every symmetric matrix is diagonalizable, the proof of this general theorem is not so easy, but the Exercise asks to prove that a 2 \times 2 symmetric matrix is always diagonalizable. Let’s suppose that A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},
\begin{align*}
\det(A - \lambda I) &= \det \begin{pmatrix} a & b \\ b & c \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det \begin{pmatrix} a - \lambda & b \\ b & c - \lambda \end{pmatrix} = (\lambda - a)(\lambda - c) - b^2 = 0.
\end{align*}

The equation
\[ \lambda^2 - (a + c)\lambda + (ac - b^2) = 0 \]
has two distinct roots if and only if the discriminant is strictly greater than zero. We have:
\[ \Delta = (a + c)^2 - 4(ac - b^2) = a^2 + 2ac + c^2 - 4ac + 4b^2 = a^2 - 2ac + c^2 + 4b^2 = (a - c)^2 + (2b)^2. \]
Since this is a sum of squares, it is nonnegative. Since \( b \neq 0 \), it is actually positive. This proves that the characteristic polynomial has two distinct real roots, so the symmetric matrix A has two distinct eigenvalues, thus it is diagonalizable.

Let’s now consider l : \mathbb{R}^2 \to \mathbb{R}^2, l(x, y) = (2x + 2y, 2x + 5y).

Observation. By definition, a linear transformation l : V \to V, V Euclidean space, is symmetric if \( \langle l(v), w \rangle = \langle v, l(w) \rangle \), for every \( v, w \in V \). We prove that a linear transformation l : \mathbb{R}^n \to \mathbb{R}^n is symmetric if and only if the matrix A(l) associated to l with respect to the standard basis of IR^n is symmetric.

We consider the ordered canonical basis of \( \mathbb{R}^2, C = \{(1, 0), (0, 1)\} \), and we will find the matrix A = A(l, C, C) associated to the given linear transformation with respect to C. For this purpose, we need to evaluate:
\[ l(1, 0) = (2, 2) \]
\[ l(0, 1) = (2, 5). \]

The matrix associated to l with respect to the canonical basis of \( \mathbb{R}^2 \) is symmetric:
\[ A = A(l, C, C) = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}. \]
This implies that \( l \) is symmetric. To diagonalize \( A \), we start by solving the characteristic equation:

\[
\det(A - \lambda I) = \det \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 2 - \lambda & 2 \\ 2 & 5 - \lambda \end{pmatrix} = (\lambda - 2)(\lambda - 5) - 2^2 = 0.
\]

Incidentally, we verify that, for a symmetric \( 2 \times 2 \) matrix \( A \), the characteristic equation has the form

\[
\lambda^2 - \text{tr} A \lambda + \det A = 0,
\]

indeed, we get:

\[
\lambda^2 - 7\lambda + 6 = 0.
\]

The eigenvalues are \( \lambda_1 = 6 \) and \( \lambda_2 = 1 \). We now look for the corresponding eigenvectors. For \( \lambda_1 = 6 \), we must find the nontrivial solutions of the system:

\[
(A - \lambda_1 I) \mathbf{v} = \mathbf{0}.
\]

The augmented matrix of this homogeneous system is

\[
\begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \end{pmatrix} R_1 + 2R_2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix}.
\]

All the eigenvectors corresponding to \( \lambda_1 = 6 \) are \( \{t(1, 2) : t \in \mathbb{R} \setminus \{0\} \} \), and the eigenspace corresponding to \( \lambda_1 = 6 \) is \( U_6 = \text{span}\{\{1, 2\}\} \).

To find the eigenvectors corresponding to the second eigenvalue \( \lambda_2 = 1 \), it is enough to recall that, for symmetric matrices, eigenvectors corresponding to distinct eigenvalues are orthogonal, thus the eigenspace corresponding to \( \lambda_2 = 1 \) is \( U_1 = \text{span}\{\{2, -1\}\} \).

A basis of \( \mathbb{R}^2 \) made of orthonormal eigenvectors of \( A \) is

\[
B = \left\{ \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right), \left( \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right) \right\}.
\]

The matrix \( P \) formed by the vectors in \( B \), written as columns, is unitary, and has the property that \( P^{-1} = P^T \):

\[
P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.
\]

Moreover,

\[
P^T A P = D = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}.
\]

7.25 Verify that the linear transformation \( l : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), defined by \( l(x, y, z) = (x + 3y + 4z, 3x + y, 4x + z) \) is symmetric, prove that it is diagonalizable and there exists a basis of \( \mathbb{R}^3 \) made of three orthogonal eigenvectors of \( l \).

**Solution.** We consider the ordered canonical basis of \( \mathbb{R}^3 \), \( C = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \), and we will find the matrix \( A = A(l, C) \) associated to the given linear transformation with respect to \( C \). For this purpose, we need to evaluate:

\[
l(1, 0, 0) = (1, 3, 4) \\
l(0, 1, 0) = (3, 1, 0) \\
l(0, 0, 1) = (4, 0, 1).
\]

Now we can write \( A = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \). The matrix associated to \( l \) with respect to the canonical basis of \( \mathbb{R}^3 \) is symmetric, this implies that \( l \) is symmetric.

To diagonalize \( A \), we start by solving the characteristic equation:

\[
\det(A - \lambda I) = \det \begin{pmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 - \lambda & 3 & 4 \\ 3 & 1 - \lambda & 0 \\ 4 & 0 & 1 - \lambda \end{pmatrix} =
\]

\[
=(1 - \lambda)^3 - 16(1 - \lambda) - 9(1 - \lambda) = (1 - \lambda)[(1 - \lambda)^2 - 25] = (1 - \lambda)(-4 - \lambda)(6 - \lambda) = 0.
\]

}\]
The characteristic equation of $\mathbf{A}$ has three distinct solutions $\lambda = 1$, $\lambda = -4$, $\lambda = 6$, with algebraic multiplicity 1. In this case, we may already conclude that $\mathbf{A}$ is diagonalizable, indeed, corresponding to each eigenvalue we will choose an eigenvector, and, since the eigenvalues are distinct, the chosen eigenvectors are linearly independent and form a basis of $\mathbb{R}^4$. We will verify that, being $\mathbf{A}$ symmetric, the found basis is indeed formed by orthogonal vectors.

Thus, we look for the eigenvectors corresponding to $\lambda = 1$. We must find the nontrivial solutions of the homogeneous system $(\mathbf{A} - \lambda \mathbf{I}) \mathbf{v} = \mathbf{0}$, where $\lambda = 1$. We solve the homogeneous system with augmented matrix:

$$
\begin{pmatrix}
0 & 3 & 4 & 0 \\
3 & 0 & 0 & 0 \\
4 & 0 & 0 & 0
\end{pmatrix}.
$$

All the eigenvectors corresponding to $\lambda = 1$ are $\{t(0,4,-3) : t \in \mathbb{R} \setminus \{0\}\}$ and the eigenspace associated to $\lambda = 1$ is $U_1 = \text{span}\{(0,4,-3)\}$.

We consider $\lambda = -4$, and find the corresponding eigenvectors solving the system:

$$
\begin{pmatrix}
5 & 3 & 4 & 0 \\
3 & 0 & 0 & 0 \\
4 & 0 & 5 & 0
\end{pmatrix} \sim
\begin{pmatrix}
3R_1 - 5R_2 & (5 & 3 & 4 & 0) \\
0 & -16 & 12 & 0 \\
0 & 12 & -9 & 0
\end{pmatrix} \sim
\begin{pmatrix}
R_2/4 & (5 & 3 & 4 & 0) \\
0 & -4 & 3 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

It is easy to deduce that all the eigenvectors corresponding to $\lambda = -4$ are $\{t(-5,3,4) : t \in \mathbb{R} \setminus \{0\}\}$ and the eigenspace associated to $\lambda = -4$ is $U_{-4} = \text{span}\{(-5,3,4)\}$.

Finally, we consider the eigenvalue $\lambda = 6$, and find the corresponding eigenvectors solving the system:

$$
\begin{pmatrix}
5 & 3 & 4 & 0 \\
3 & 0 & 0 & 0 \\
4 & 0 & -5 & 0
\end{pmatrix} \sim
\begin{pmatrix}
3R_1 + 5R_2 & (5 & 3 & 4 & 0) \\
0 & -16 & 12 & 0 \\
0 & 12 & -9 & 0
\end{pmatrix} \sim
\begin{pmatrix}
R_2/4 & (5 & 3 & 4 & 0) \\
0 & -4 & 3 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

The eigenvectors corresponding to $\lambda = 6$ are $\{t(5,3,4) : t \in \mathbb{R} \setminus \{0\}\}$ and the eigenspace associated to $\lambda = 6$ is $U_6 = \text{span}\{(5,3,4)\}$.

A basis of $\mathbb{R}^3$ made of eigenvectors of $l$ is $B = \{(0,4,-3), (-5,3,4), (5,3,4)\}$. We easily verify that the found eigenvectors are orthogonal:

$$(0,4,-3) \cdot (-5,3,4) = 0 + 12 - 12 = 0,$$

$$(0,4,-3) \cdot (5,3,4) = 0 + 12 - 12 = 0,$$

$$( -5,3,4) \cdot (5,3,4) = -25 + 9 + 16 = 0.$$

We conclude that the given matrix $\mathbf{A}$ is similar to a diagonal matrix, and the matrix $\mathbf{P} = \begin{pmatrix} 0 & -5 & 5 \\ 4 & 3 & 3 \\ -3 & 4 & 4 \end{pmatrix}$ is such that

$$
\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{pmatrix}.
$$

If we normalise the vectors in $B$, and create a new basis $\tilde{B}$ of $\mathbb{R}^3$ made of orthonormal eigenvectors of $\mathbf{A}$,

$$
\tilde{B} = \left\{ \frac{1}{5} (0,4,-3), \frac{1}{5\sqrt{2}} (-5,3,4), \frac{1}{5\sqrt{2}} (5,3,4) \right\},
$$

the matrix $\tilde{\mathbf{P}}$, made of vectors in $\tilde{B}$ written as columns,

$$
\tilde{\mathbf{P}} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\
\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}}
\end{pmatrix}
$$

is unitary, has the property that $\tilde{\mathbf{P}}^{-1} = \tilde{\mathbf{P}}^T$, thus:

$$
\tilde{\mathbf{P}}^{-1} \mathbf{A} \tilde{\mathbf{P}} = \tilde{\mathbf{P}}^T \mathbf{A} \tilde{\mathbf{P}} = \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{pmatrix}.
$$

7.27) Identify the curve with equation $12x^2 - 12xy + 7y^2 + 60x - 38y + 31 = 0$. 

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Solution. We will solve this exercise following a general approach, see for example the lecture notes *Elementary Linear Algebra*, by K.R. Matthews, Department of Mathematics, University of Queensland.

Given the curve with equation $ax^2 + 2hxy + by^2 + cx + dy + e = 0$, the expression $ax^2 + 2hxy + by^2$ is called a quadratic form in $x$ and $y$. Using matrices, we have the identity

$$ax^2 + 2hxy + by^2 = (x \ y) \begin{pmatrix} a & h \\ h & b \end{pmatrix} (x \ y) = X^TAX,$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$. $A$ is called the matrix of the quadratic form, and it is a symmetric $2 \times 2$ real matrix. It follows that $A$ is diagonalizable and there exists a matrix $P$ made of orthonormal eigenvectors of $A$ such that

$$P^TAP = D$$

is a diagonal matrix with nonzero diagonal entries $\lambda_1$ and $\lambda_2$, the real eigenvalues of $A$. The quadratic form satisfies the following:

$$ax^2 + 2hxy + by^2 = X^TAX = X^T(PDP^T)X = (P^TX)^TDP^TX,$$

and if $Y = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = P^TX$, then $X = PY$ and

$$ax^2 + 2hxy + by^2 = Y^TDY = (x_1 \ y_1) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (x_1 \ y_1) = \lambda_1 x_1^2 + \lambda_2 y_1^2,$$

that expresses the fact that the transformation (change of basis)

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}$$

that can be shown to be a rotation of axes, diagonalize the quadratic form corresponding to $A$.

Then, under the rotation $X = PY$, the original quadratic equation becomes

$$\lambda_1 x_1^2 + \lambda_2 y_1^2 + cx_1 + dy_1 + e = 0.$$

Completing the square in $x_1$ and $y_1$, after a suitable transformation (a translation of axes)

$$x_2 = x_1 + \alpha, \quad y_2 = y_1 + \beta$$

the original equation will reduce to

$$ax_2^2 + by_2^2 = w, \quad u, v, w \in \mathbb{R}, \text{ not all zero},$$

from which we will be able to classify the original curve.

Let’s work out the detailed computation for the given curve $12x^2 - 12xy + 7y^2 + 60x - 38y + 31 = 0$. In this case,

$$A = \begin{pmatrix} 12 & -6 \\ -6 & 7 \end{pmatrix}.$$

The characteristic equation of $A$ is $\lambda^2 - (\text{tr} A)\lambda + \text{det} A = 0$, or

$$\lambda^2 - 19\lambda + 48 = (\lambda - 16)(\lambda - 3) = 0.$$

$A$ has distinct eigenvalues $\lambda_1 = 16$ and $\lambda_2 = 3$.

For $\lambda_1 = 16$, we look for corresponding eigenvector solving $(A - \lambda_1 I)v = 0$, thus we solve the homogeneous system:

$$\begin{pmatrix} -4 & -6 & 0 \\ -6 & -9 & 0 \end{pmatrix} \begin{pmatrix} -R_1/2 \\ R_1/2 - R_3/3 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

All the eigenvectors corresponding to $\lambda_1 = 16$ are $\{t(3, -2) : t \in \mathbb{R} \setminus \{0\}\}$, and the eigenspace corresponding to $\lambda_1 = 16$ is $U_{16} = \text{span}\{(3, -2)\}$.

To find the eigenvectors corresponding to the second eigenvalue $\lambda_2 = 3$, it is enough to recall that, for symmetric matrices, eigenvectors corresponding to distinct eigenvalues are orthogonal, thus the eigenspace corresponding to $\lambda_2 = 3$ is $U_3 = \text{span}\{(2, 3)\}$.  

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We normalize the found eigenvectors and create a basis $B$ of $\mathbb{R}^2$ made of orthonormal eigenvectors of $A$, $B = \left\{ \left( \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \right), \left( \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right) \right\}$. Using $B$, we create the unitary matrix

$$P = \begin{pmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}$$

that has the property that $P^{-1} = P^T$, and $P^TAP = D = \begin{pmatrix} 16 & 0 \\ 0 & 3 \end{pmatrix}$. Moreover, the fact that $\det P = 1$ guarantees that the rotation $X = PY$ preserves the orientation of the axes. More explicitly the transformation

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = PY = P \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

can be written as:

$$x = \frac{3x_1 + 2y_1}{\sqrt{13}}, \quad y = \frac{-2x_1 + 3y_1}{\sqrt{13}}.$$

Under the rotation $X = PY$, the original quadratic equation becomes

$$16x_1^2 + 3y_1^2 + \frac{60}{\sqrt{13}}(3x_1 + 2y_1) - \frac{38}{\sqrt{13}}(-2x_1 + 3y_1) + 31 = 0$$

or

$$16x_1^2 + 3y_1^2 + \frac{256}{\sqrt{13}}x_1 + \frac{6}{\sqrt{13}}y_1 + 31 = 0.$$

We now complete the square in $x_1$ and $y_1$:

$$16\left( x_1^2 + \frac{16}{\sqrt{13}}x_1 \right) + 3\left( y_1^2 + \frac{2}{\sqrt{13}}y_1 \right) + 31 = 0$$

$$16\left( x_1 + \frac{8}{\sqrt{13}} \right)^2 + 3\left( y_1 + \frac{1}{\sqrt{13}} \right)^2 = 48.$$

Now, if we perform a translation of axes to the new origin $(x_1, y_1) = (-\frac{8}{\sqrt{13}}, \frac{1}{\sqrt{13}})$:

$$x_2 = x_1 + \frac{8}{\sqrt{13}}, \quad y_2 = y_1 + \frac{1}{\sqrt{13}}$$

the quadratic equation reduces to

$$16x_2^2 + 3y_2^2 = 48$$

or

$$\frac{x_2^2}{3} + \frac{y_2^2}{16} = 1.$$

This equation is now in standard form and represent an ellipse centred at $(x_2, y_2) = (0, 0)$ and with axes of symmetry the $x_2$ and $y_2$ axes. In terms of the original $x, y$ coordinates, knowing that $(x_1, y_1) = (-\frac{8}{\sqrt{13}}, \frac{1}{\sqrt{13}})$, we find that the center is

$$x = \frac{3x_1 + 2y_1}{\sqrt{13}} = -2, \quad y = \frac{-2x_1 + 3y_1}{\sqrt{13}} = 1.$$

Also, from $(x, y) = \left( \frac{3x_1 + 2y_1}{\sqrt{13}}, \frac{-2x_1 + 3y_1}{\sqrt{13}} \right)$, we deduce $(x_1, y_1) = \left( \frac{3x-2y}{\sqrt{13}}, \frac{2x+3y}{\sqrt{13}} \right)$. Thus the $y_2$-axis is given by

$$0 = x_2 = x_1 + \frac{8}{\sqrt{13}} = \frac{3x - 2y}{\sqrt{13}} + \frac{8}{\sqrt{13}}$$

or $3x - 2y + 8 = 0$. Similarly, the $x_2$-axis is given by

$$0 = y_2 = y_1 + \frac{1}{\sqrt{13}} = \frac{2x + 3y}{\sqrt{13}} + \frac{1}{\sqrt{13}}$$

or $2x + 3y + 1 = 0$.  

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8.1) Find the parametric and canonical equation of the line \( p \) passing through the points \( A = [1, 0, 2] \) and \( B = [3, 1, -2] \); check whether the point \( M = [7, 3, 1] \) lies on \( p \).

8.2) Verify that the line \( p \) lies on the plane \( \sigma \), where:

\[
p : X = [1, 0, 1] + t(2, 1, 3), \quad t \in \mathbb{R}, \quad \sigma : 2x - 2y - z - 1 = 0.
\]

8.3) Find the equation of the planes \( \rho \) and \( \sigma \), verify that they are not parallel and find the parametric equation of the line \( p \), intersection of \( \rho \) and \( \sigma \), where:

- \( \rho \) is the plane containing the point \( M = [1, -2, 3] \) and orthogonal to the vector \( \mathbf{n} = (4, 5, -6) \);
- \( \sigma \) is the plane passing through the points \( A = [2, 5, -1] \), \( B = [2, -3, 3] \), and \( C = [4, 5, 0] \).

8.4) Given the plane \( \sigma \) and the point \( A \), verify that \( A \) does not belong to \( \sigma \) and find the equation of the plane \( \rho \) through \( A \) parallel to \( \sigma \), where:

\[
\sigma : 2x + 7y - 3z = 1 \quad A = [1, 1, 1].
\]

8.5) Find the angle between the planes \( \rho \) and \( \sigma \), if \( \rho \) passes through the points \( M_1 = [-2, 2, 2] \), \( M_2 = [0, 5, 3] \) and \( M_3 = [-2, 3, 4] \), and \( \sigma \) has equation \( 3x - 4y + z + 5 = 0 \).

8.6) Find the angle between the lines \( p \) and \( q \), if:

\[
p : X = [1, 2, 1] + t(1, -2, -3), \quad t \in \mathbb{R}, \quad q : \text{contains the points} \ A = [2, -3, 1], \ B = [-3, 0, 2].
\]

8.7) Find the distance between the point \( A = [8, -7, 1] \) and the plane with equation \( 2x + 3y - 4z + 5 = 0 \).

8.8) Find the distance between the point \( A = [-1, -1, 1] \) and the line \( p \), intersection of the planes \( 2x - y + 3z + 2 = 0 \), and \( x + 2y - z + 1 = 0 \).

8.9) Given \( A = [2, 9, 8], B = [6, 4, -2] \) and \( C = [7, 15, 7] \), show that \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) are perpendicular, then find \( D \) so that \( ABCD \) forms a rectangle.

8.10) Given the line \( p \), \( X = [1, 2, 3] + t(1, 2, -2) \) and the point \( A = [3, 3, 5] \), find the points on \( p \) that have distance \( 3\sqrt{5} \) from \( A \).

8.11) Given the line \( p \) passing through the points \( A = [1, 2, 1] \) and \( B = [2, -1, 3] \), find the point \( P \) on \( p \) closest to the origin and the shortest distance from the origin to \( p \).

8.12) Find the plane \( \sigma \) going through the points \( A = [2, 1, -3], B = [-3, 2, 0], C = [0, 3, -5] \) and determine the angle between \( \sigma \) and the line \( p : \frac{x - 2}{-1} = \frac{y + 1}{1} = \frac{z - 3}{2} \).

8.13) Show that the planes \( x + y - 2z = 1 \) and \( x + 3y - z = 4 \) intersect in a line and find the distance between the point \( C = [1, 0, 1] \) and this line.

8.14) Find the line \( p \) going through the points \( A = [2, 1, 0] \) and \( B = [1, -3, 2] \) and the plane \( \sigma \) going through \( P = [4, 1, 1] \) that has the properties that \( d(P, p) = d(p, \sigma) \), where \( d \) indicates the distance.

8.15) Find an equation for the plane through \( P = [1, 0, 1] \) and containing the line of intersection of the planes \( x + y + 2z = 1 \) and \( x + 3y + z = 4 \).

8.16) Find an equation for the plane passing through the point \( A = [1, 0, -2] \) and containing the line \( p \) with vector equation \( X = [1, 1, -1] + t(3, 2, 0), \quad t \in \mathbb{R} \).

8.17) Find an equation for the plane passing through \( P = [6, 0, 2] \) and perpendicular to the line of intersection of the planes \( x + y - 2z = 4 \) and \( 3x - 2y + z = 1 \).
8.18) Find the intersection of the line \( p : X = [1, -2, 0] + t(3, 4, 2) \) \( t \in \mathbb{R} \) with the plane \( \sigma \) containing the point \( S = [19, 13, 9] \) and the line \( q : X = [-3, -1, 2] + u(2, 0, 3), \ u \in \mathbb{R} \).

8.19) Prove that the lines \( p \) and \( q \) are skewed, find their distance \( d(p, q) \), and find also the points \( P \in p, Q \in q \) such that \( d(p, q) = d(P, Q) \), where
\[
p : X = [1, 2, 1] + t(5, -2, -3), \ t \in \mathbb{R}, \quad q : X = [-2, 6, 3] + s(4, -3, -1), \ s \in \mathbb{R}.
\]

8.20) Given the line \( p \) through \( A = [1, 2, 1] \) and \( B = [3, -1, 2] \) and the line \( q \) through \( C = [1, 0, 2] \) and \( D = [2, 1, 3] \), prove that the distance between \( p \) and \( q \) is \( \frac{16}{\sqrt{62}} \).

8.21) Given the line \( p \) and two points \( A \) and \( B \), find a point \( M \) on \( p \) that is equally far away from \( A \) and \( B \), where:
\[
p = \rho \cap \sigma, \ \rho : x + y - z - 3 = 0, \ \sigma : 3y - 2z + 1 = 0, \quad A = [1, 0, -1], \ \text{and} \ B = [3, 4, 5].
\]

8.22) Prove that the lines \( p \) and \( q \) are skewed, and find an equation of the line \( r \) that intersects \( p \) and \( q \) and is orthogonal to both, where:
\[
p : X = [-4, 4, 1] + t(2, -1, -2), \ t \in \mathbb{R}, \quad q : X = [-5, 5, 5] + s(4, -3, -5), \ s \in \mathbb{R}.
\]

8.23) Find the point \( R \) symmetric to \( P = [-4, 5, 8] \) with respect to the line \( p \) through \( A = [9, 4, 10] \) and \( B = [-6, 1, 1] \).

8.24) Find the point \( R \) symmetric to \( P = [-1, 5, 0] \) with respect to the plane \( \sigma \) through \( A = [4, 2, 1], B = [-4, -3, -1] \) and \( C = [0, 1, 3] \).

8.25) A line with direction vector \( v = (0, 9, -1) \) intersects lines \( p \) and \( q \), find the coordinates of the points of intersection, where
\[
p : \frac{x - 8}{5} = \frac{y - 5}{1} = \frac{z}{-1}, \quad \text{and} \quad q : \frac{x}{1} = \frac{y - 1}{-2} = \frac{z + 1}{1}.
\]

8.26) Find the line \( p \) parallel to \( s : \frac{x - 3}{1} = \frac{y + 1}{3} = \frac{z - 3}{-2} \) that intersects the lines
\[
q : \frac{x - 3}{3} = \frac{y - 1}{-7} = \frac{z - 2}{1}, \quad \text{and} \quad r : X = [1, 4, 4] + t(3, 2, -1), \ t \in \mathbb{R}.
\]

8.27) Find the distance of the point \( A = [3, 2, 1] \) from the plane containing the lines \( p \) and \( q \), where
\[
p : \frac{x + 1}{1} = \frac{y - 3}{2} = \frac{z - 2}{2}, \quad \text{and} \quad q : X = [0, 4, 2] + t(1, 1, 0), \ t \in \mathbb{R}.
\]

8.28) Find the plane going through the point \( A = [2, -1, 1] \) and orthogonal to the line \( p \) of intersection of the planes
\[
\sigma : X = [0, 4, 2] + t(1, 1, 0) + u(1, 1, 2), \ t, u \in \mathbb{R}, \quad \rho : X = [1, -1, 1] + s(-1, 2, 2) + v(3, -4, 1), \ s, v \in \mathbb{R}.
\]
Solutions of odd number problems - Analytic geometry in three dimensions

8.1) Find the parametric and the vector equation of the line \( p \) passing through the points \( A = [1, 0, 2] \) and \( B = [3, 1, -2] \); check whether the point \( M = [7, 3, 1] \) lies on \( p \).

**Solution.** We recall that a line \( p \), passing through a point \( A = [a, b, c] \), with direction vector \( \mathbf{v} = (v_1, v_2, v_3) \), has vector equation:

\[
X = A + t\mathbf{v}, \quad t \in \mathbb{R},
\]

and consequently (fixing \( X = [x, y, z] \)) parametric equations:

\[
\begin{align*}
x &= a + tv_1 \\
y &= b + tv_2 \\
z &= c + tv_3,
\end{align*}
\]

In our case, \( A = [1, 0, 2] \) is given, and we may evaluate \( \mathbf{v} = B - A = [3, 1, -2] - [1, 0, 2] = (2, 1, -4) \). Thus \( X = [1, 0, 2] + t(2, 1, -4), \quad t \in \mathbb{R} \) is the required vector equation of \( p \), and its parametric equation is:

\[
\begin{align*}
x &= 1 + 2t \\
y &= t \\
z &= 2 - 4t, \quad t \in \mathbb{R}.
\end{align*}
\]

To verify if the point \( M = [7, 3, 1] \) lies on \( p \), it is enough to substitute the coordinates of \( M \) in one of the equations of \( p \) and verify if there exists a value of \( t \in \mathbb{R} \) for which the equation(s) is(are) satisfied. We substitute \( [7, 3, 1] = [x, y, z] \) into the parametric equation of \( p \), and get:

\[
\begin{align*}
7 &= 1 + 2t \\
3 &= t \\
1 &= 2 - 4t, \quad t \in \mathbb{R},
\end{align*}
\]

This is equivalent to:

\[
\begin{align*}
6 &= 2t \\
3 &= t \\
1 &= 4t, \quad t \in \mathbb{R},
\end{align*}
\]

the value \( t = 3 \) satisfies the first two equations but not the third, thus there exists no value of \( t \) that solves the given system, \( M = [7, 3, 1] \) does not lie on \( p \).

**Observation.** An equivalent result is obtained with the following reasoning: the point \( M \) belongs to line \( p \) if and only if the vector \( \overrightarrow{AM} \) is parallel to the vector \( \overrightarrow{AB} \), i.e. if \( \overrightarrow{AM} = M - A = [7, 3, 1] - [1, 0, 2] = (6, 3, -1) \) is a multiple of the vector \( \overrightarrow{AB} = B - A = [3, 1, -2] - [1, 0, 2] = (2, 1, -4) \). This is not the case, thus \( M \) does not belong to \( p \).

8.3) Find the equation of the planes \( \rho \) and \( \sigma \), verify that they are not parallel and find the parametric equation of the line \( p \), intersection of \( \rho \) and \( \sigma \), where:

\( \rho \) is the plane containing the point \( M = [1, -2, 3] \) and orthogonal to the vector \( \mathbf{n} = (4, 5, -6) \);

\( \sigma \) is the plane passing through the points \( A = [2, 5, -1], B = [2, -3, 3] \), and \( C = [4, 5, 0] \).

**Solution.** The point-normal equation of a plane containing a point \( A = [x_0, y_0, z_0] \) and orthogonal to a vector \( \mathbf{n} = (a, b, c) \) is obtained posing the condition that, if \( X = [x, y, z] \) is any other point of the plane, then the vector \( \overrightarrow{AX} \) must be orthogonal to \( \mathbf{n} \). Thus, using the scalar product:

\[
\overrightarrow{AX} \cdot \mathbf{n} = 0, \quad \Leftrightarrow \quad (X - A) \cdot \mathbf{n} = 0 \quad \Leftrightarrow \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.
\]

For the plane \( \rho \), we get

\[
\rho : \quad \overrightarrow{MX} \cdot \mathbf{n} = 0, \quad \Leftrightarrow \quad ([x, y, z] - [1, -2, 3]) \cdot (4, 5, -6) = 0 \quad \Leftrightarrow \quad 4x + 5y - 6z + 24 = 0.
\]
To find the equation of the plane $\sigma$, we first need to identify a vector $m$ orthogonal to $\sigma$. Knowing that the points $A, B$, and $C$ lie on $\sigma$, we consider the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and create $m = \overrightarrow{AB} \times \overrightarrow{AC}$. We now evaluate:

$$\overrightarrow{AB} = B - A = [2, -3, 3] - [2, 5, -1] = (0, -8, 4),$$
$$\overrightarrow{AC} = C - A = [4, 5, 0] - [2, 5, -1] = (2, 0, 1),$$

$$m = \overrightarrow{AB} \times \overrightarrow{AC} = (0, -8, 4) \times (2, 0, 1) = \begin{pmatrix} 1 & j & k \\ 0 & -8 & 4 \\ 2 & 0 & 1 \end{pmatrix} = (-8, 8, 16) = -8(1, -1, -2).$$

We choose to use the shorter vector $m_1 = (1, -1, -2)$, still orthogonal to the plane $\sigma$, and the equation of $\sigma$ is obtained:

$$\sigma: \overrightarrow{AX} \cdot m_1 = 0, \quad \Leftrightarrow \quad ([x, y, z] - [2, 5, -1]) \cdot (1, -1, -2) = 0 \quad \Leftrightarrow \quad x - y - 2z + 1 = 0.$$

To verify that $\rho$ and $\sigma$ are not parallel, it is enough to observe that the vectors $n = (4, 5, -6)$ and $m_1 = (1, -1, -2)$ are not parallel, since they are not proportional. We now need to find the equation of the line $p = \rho \cap \sigma$. The direction vector $v$ of $p$ can be obtained as the cross product $n \times m_1$, because $v$ lies on both $\rho$ and $\sigma$, thus must be orthogonal to both $n$ and $m_1$:

$$v = n \times m_1 = \begin{vmatrix} i & j & k \\ 4 & 5 & -6 \\ 1 & -1 & -2 \end{vmatrix} = (-16, 2, -9).$$

We now need a point $P$ belonging to $\rho \cap \sigma$, thus we look for a solution of the system:

$$\rho: \begin{cases} 4x + 5y - 6z + 24 = 0 \quad & x - y - 2z + 1 = 0 \end{cases}$$

$$\sigma: \begin{cases} x - y - 2z + 1 = 0 \quad & 9y + 2z + 20 = 0 \end{cases},$$

and the point $P = [x_P, y_P, z_P]$ can be easily guessed starting from the last equation, posing $y_p = 0$, we get $z_p = -10$, and, substituting these values into the first equation, we get $x_p = -21$, thus $P = [-21, 0, -10]$. Therefore:

$$p: \quad X = P + tv, \quad t \in \mathbb{R} \quad \Leftrightarrow \quad X = [-21, 0, -10] + t(-16, 2, -9), \quad t \in \mathbb{R}.$$

**Observation.** The equation of $p$ is given by all solutions of the system formed with the equations of the intersecting planes: $[-21, 0, -10]$ is a particular solution, and $\{(-16, 2, -9), \quad t \in \mathbb{R}\}$ are all solutions of the homogeneous system.

8.5) Find the angle between the planes $\rho$ and $\sigma$, if $\rho$ passes through the points $M_1 = [2, 2, 2]$, $M_2 = [0, 5, 3]$ and $M_3 = [-2, 3, 4]$, and $\sigma$ has equation $3x - 4y + z + 5 = 0$.

**Solution.** The angle between two planes is the same as the angle between the orthogonal vectors to the planes. For $\sigma$ we can deduce from its equation that $n_\sigma = (3, -4, 1)$. For $\rho$, we obtain $n_\rho = M_1M_2 \times M_1M_3$

$$n_\rho = (M_2 - M_1) \times (M_3 - M_1) = ([0, 5, 3] - [-2, 2, 2]) \times ([2, 3, 4] - [-2, 2, 2]) =$$
$$= (2, 3, 1) \times (0, 1, 2) = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (5, -4, 2).$$

Thus, the cosine of the required angle $\theta$ is

$$\cos \theta = \frac{n_\rho \cdot n_\sigma}{\|n_\rho\| \cdot \|n_\sigma\|} = \frac{(3, -4, 1) \cdot (5, -4, 2)}{\|(3, -4, 1)\| \cdot \|(5, -4, 2)\|} = \frac{33}{\sqrt{26} \sqrt{45}} = \frac{11}{15 \sqrt{10}}.$$

8.7) Find the distance between the point $A = [8, -7, 1]$ and the plane $\sigma$ with equation $2x + 3y - 4z + 5 = 0$.

**Solution.** We may solve the problem in several different ways, we will present a first short and elegant solution that relies on the properties of inner product, and a second more geometrical and standard way.

**First method.** Let’s consider the vector $n = (2, 3, -4)$ orthogonal to the plane and any point $Q$ on the plane, for example $Q = (-1, -1, 0)$, that we can guess from the equation of $\sigma$. From the properties of inner product (given any two vectors $v, u$, we have $u \cdot v = \|v\| \cdot \|u\| \cos \theta$, where $\theta$ is the angle in between $u$ and
v), the vector $\overrightarrow{AQ}$ is such that its projection on the unit vector $\frac{n}{\|n\|}$ has length equal to the distance $d(A, \sigma)$ of $A$ from the plane $\sigma$. Thus

$$d(A, \sigma) = \left\| (Q - A) \cdot \frac{n}{\|n\|} \right\| = \frac{|(9, -6, 1) \cdot (2, 3, -4)|}{\|n\|} = \frac{|-4|}{\sqrt{29}} = \frac{4}{\sqrt{29}}.$$

**Second method.** Let’s find the projection $A'$ of the point $A$ on the plane $\sigma$, then $d(A, \sigma) = \|\overrightarrow{AA'}\|$. We first look for the equation of the line $p$ going through $A$ and orthogonal to the plane, thus $p$ has direction vector $n = (2, 3, -4)$.

$$p : \quad X = A + t \cdot n, \quad t \in \mathbb{R} \quad \Rightarrow \quad X = [8, -7, 1] + t(2, 3, -4), \quad t \in \mathbb{R}.$$

Now we find $A' = p \cap \sigma$. We take the parametric equation of $p$, $[x, y, z] = [8 + 2t, -7 + 3t, 1 - 4t]$, and substitute it into the equation of $\sigma$, to find the particular value ($t_0$) of $t$ that corresponds to the point $A'$.

$$2x + 3y - 4z + 5 = 0 \quad \Rightarrow \quad 2(8 + 2t) + 3(-7 + 3t) - 4(1 - 4t) + 5 = 0 \quad \Rightarrow \quad t_0 = -\frac{4}{29}.$$

Therefore $A' = [8, -7, 1] - \frac{4}{29}(2, 3, -4)$, and

$$d(A, \sigma) = \|\overrightarrow{AA'}\| = \|A - A'\| = \frac{4}{\sqrt{29}} \|(2, 3, -4)\| = \frac{4}{\sqrt{29}}.$$

8.9) Given $A = [2, 9, 8]$, $B = [6, 4, -2]$ and $C = [7, 15, 7]$, show that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular, then find $D$ so that $ABCD$ forms a rectangle.

**Solution.** Let’s evaluate: $\overrightarrow{AB} = B - A = [6, 4, -2] - [2, 9, 8] = (4, -5, -10)$, $\overrightarrow{AC} = C - A = [7, 15, 7] - [2, 9, 8] = (5, 6, -1)$, thus $\overrightarrow{AB} \cdot \overrightarrow{AC} = (4, -5, -10) \cdot (5, 6, -1) = 0$ implies that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular. Now we will find $D$ just posing the condition that the middle point $M$ of the segment $CB$, that is one diagonal of the rectangle $ABCD$, must also be the middle point of the other diagonal of the rectangle $AD$. Thus:

$$M = \frac{1}{2}(B + C) = \frac{1}{2}([6, 4, -2] + [7, 15, 7]) = \frac{1}{2}([13, 19, 5]),$$

and, since $\frac{1}{2}(D + A) = M$ implies $D = 2M - A$, we get

$$D = 2 \cdot \frac{1}{2} [13, 19, 5] - [2, 9, 8] = [11, 10, -3].$$

8.11) Given the line $p$ passing through the points $A = [1, 2, 1]$ and $B = [2, -1, 3]$, find the point $P$ on $p$ closest to the origin and the shortest distance from the origin to $p$.

**Solution.** To find the vector equation of $p$, we first calculate the direction vector $v$ of $p$: $v = B - A = [2, -1, 3] - [1, 2, 1] = (1, -3, 2)$. Now we can write

$$p : \quad X = A + t \cdot v, \quad t \in \mathbb{R} \quad \Rightarrow \quad X = [1, 2, 1] + t(1, -3, 2), \quad t \in \mathbb{R}.$$

If $X$ is a general point of the line $p$, its distance from the origin is:

$$d(X, 0) = \|X - 0\| = \| (1 + t, 2 - 3t, 1 + 2t) \| = \sqrt{(1 + t)^2 + (2 - 3t)^2 + (1 + 2t)^2} = \sqrt{1 + 2t + t^2 + 4 - 12t + 9t^2 + 1 + 4t + 4t^2} = \sqrt{14t^2 - 6t + 6}.$$

The value $t_0$ that correspond to the minimum value of the distance is the critical point of the function $f(t) = \sqrt{14t^2 - 6t + 6}$, or also the critical point of the function that represents the distance squared: $g(t) = f^2(t) = 14t^2 - 6t + 6$, that is a parabola with vertex (and minimum) at $t_0 = \frac{1}{14}$. We conclude that the point $P$ on $p$ closest to the origin is:

$$P = [1, 2, 1] + t_0(1, -3, 2) = [1, 2, 1] + \frac{3}{14}(1, -3, 2) = \left[ \frac{17}{14}, \frac{19}{14}, \frac{20}{14} \right],$$

and its distance from the origin is:

$$\left\| \left( \frac{17}{14}, \frac{19}{14}, \frac{20}{14} \right) \right\| = f \left( \frac{3}{14} \right) = \frac{\sqrt{75}}{14}.$$
8.13) Show that the planes $x + y - 2z = 1$ and $x + 3y - z = 4$ intersect in a line and find the distance between the point $C = [1, 0, 1]$ and this line.

Solution. The plane $\rho$ with equation $x + y - 2z = 1$ has normal vector $\mathbf{n} = (1, 1, -2)$, and the plane $\sigma$ with equation $x + 3y - z = 4$ has normal vector $\mathbf{m} = (1, 3, -1)$. Since $\mathbf{n}$ and $\mathbf{m}$ are not proportional, the planes are not parallel, thus they must intersect in a line. The equation of the line can be obtained as the set of all solutions of the following system:

\[
\begin{align*}
\rho: & \quad x + y - 2z = 1 \\
\sigma: & \quad x + 3y - z = 4
\end{align*}
\]

Setting $y = t$, from the second equation we get $z = -2t + 3$, and, substituting these values into the first equation, we get $x = -5t + 7$. The equation of the line $p$ is:

\[
p: \quad [x, y, z] = [-5t + 7, t, -2t + 3], \quad t \in \mathbb{R}, \quad \text{or} \quad X = [7, 0, 3] + t(-5, 1, -2), \quad t \in \mathbb{R}.
\]

The line $p$ has direction vector $\mathbf{v} = (-5, 1, -2)$, and is passing through the point $A = [7, 0, 3]$. To find the distance between $C$ and $p$, we will look for the projection $C'$ of $C$ on $p$ and evaluate $d(C, p) = \|CC'\|$. Looking for $C'$, we search for the unique particular point $X$ of $p$, so that the vector $\overrightarrow{CX}$ is orthogonal to $\mathbf{v}$. Thus, we pose the condition

\[
\overrightarrow{CX} \cdot \mathbf{v} = 0
\]

and solve it for $t$.

\[
\overrightarrow{CX} \cdot \mathbf{v} = 0 \iff (X - C) \cdot \mathbf{v} = 0 \iff \left([-5t + 7, t, -2t + 3] - [1, 0, 1]\right) \cdot (-5, 1, -2) = 0 \iff \left(-5t + 6, -2t + 2\right) \cdot (-5, 1, -2) = 0 \iff 30t - 34 = 0 \iff t = \frac{17}{15}
\]

Therefore $t_0 = \frac{17}{15}$ is the value of $t$ that substituted into the equation of $p$ will give the point $C'$:

\[
C' = \left(-\frac{5}{15}, \frac{17}{15}, -2\frac{17}{15} + 3\right) = \left(2\frac{20}{15}, \frac{17}{15}, \frac{11}{15}\right).
\]

Thus,

\[
d(C, p) = \|\overrightarrow{CC'}\| = \|C' - C\| = \left\|\left(\frac{20}{15}, \frac{17}{15}, \frac{11}{15}\right) - (1, 0, 1)\right\| = \left\|\left(\frac{5}{15}, \frac{2}{15}, -\frac{4}{15}\right)\right\| = \frac{1}{15}\sqrt{45} = \frac{1}{\sqrt{5}}.
\]

8.15) Find an equation for the plane through $P = [1, 0, 1]$ and containing the line of intersection of the planes $x + y + 2z = 1$ and $x + 3y + z = 4$.

Solution. Let’s call $p$ the line of intersection of the planes $x + y + 2z = 1$ and $x + 3y + z = 4$. To get an equation for $p$ we solve the system:

\[
\begin{align*}
x + y + 2z &= 1 \\
x + 3y + z &= 4
\end{align*}
\]

Setting $y = t$, from the second equation we get $z = 2t - 3$, and, substituting these values into the first equation, we get $x = -5t + 7$. The equation of the line $p$ is:

\[
p: \quad [x, y, z] = [-5t + 7, t, 2t - 3], \quad t \in \mathbb{R}, \quad \text{or} \quad X = [7, 0, -3] + t(-5, 1, 2), \quad t \in \mathbb{R}.
\]

The line $p$ has direction vector $\mathbf{v} = (-5, 1, 2)$, and is passing through the point $A = [7, 0, -3]$. The plane $\sigma$ passing through $P$ and containing $p$, contains the vector $\mathbf{v}$ and also all possible vectors joining any point of $p$ (for example $A$) with $P$, thus $\sigma$ has normal vector $\mathbf{n} = \mathbf{v} \times \overrightarrow{AP}$:

\[
\mathbf{n} = \mathbf{v} \times \overrightarrow{AP} = \mathbf{v} \times (P - A) = \mathbf{v} \times ([1, 0, 1] - [7, 0, -3]) = (-5, 1, 2) \times (-6, 0, 4) =
\]

\[
= \det \begin{pmatrix} i & j & k \\ -5 & 1 & 2 \\ -6 & 0 & 4 \end{pmatrix} = (4, 8, 6) = 2(2, 4, 3).
\]

The plane $\sigma$ passing through $P = [1, 0, 1]$ and with normal vector the ”shorter” $\mathbf{n}_1 = (2, 4, 3)$, has equation:

\[
\sigma: \quad (X - P) \cdot \mathbf{n}_1 = 0, \quad \implies \quad \sigma: \quad ([x, y, z] - [1, 0, 1]) \cdot (2, 4, 3) = 0, \quad \implies \quad \sigma: \quad 2x + 4y + 3z - 5 = 0.
\]
8.17) Find an equation for the plane passing through \( P = [6, 0, 2] \) and perpendicular to the line of intersection of the planes \( x + y - 2z = 4 \) and \( 3x - 2y + z = 1 \).

**Solution.** A plane perpendicular to a line with direction vector \( \mathbf{v} \) has normal vector \( \mathbf{n} = \mathbf{v} \). Thus we need to find the direction vector of the line of intersection of the planes \( \sigma : x + y - 2z = 4 \) and \( \rho : 3x - 2y + z = 1 \). Knowing that \( \mathbf{n}_\sigma = (1, 1, -2) \), and \( \mathbf{n}_\rho = (3, -2, 1) \), we evaluate:

\[
\mathbf{n} = \mathbf{v} = \mathbf{n}_\sigma \times \mathbf{n}_\rho = (1, 1, -2) \times (3, -2, 1) = \det \begin{pmatrix}
1 & 1 & -2 \\
3 & -2 & 1
\end{pmatrix} = (-3, -7, -5).
\]

The plane \( \tau \) passing through \( P = [6, 0, 2] \) and with normal vector \( \mathbf{n}_1 = (3, 7, 5) \) (the opposite of \( \mathbf{n} \)) has equation:

\[
\tau : (X - P) \cdot \mathbf{n}_1 = 0, \quad \implies \tau : ([x, y, z] - [6, 0, 2]) \cdot (3, 7, 5) = 0, \quad \implies \tau : 3x + 7y + 5z - 28 = 0.
\]

8.19) Prove that the lines \( p \) and \( q \) are skewed, find their distance \( d(p, q) \), and find also the points \( P \in p, Q \in q \) such that \( d(p, q) = d(P, Q) \), where

\[
p : \quad X = [1, 2, 1] + t(5, -2, -3), \quad t \in \mathbb{R}, \quad q : \quad X = [-2, 6, 3] + s(4, -3, -1), \quad s \in \mathbb{R}.
\]

**Solution.** Two lines with vector equation:

\[
p : \quad X = A + t\mathbf{v}, \quad t \in \mathbb{R} \quad \text{and} \quad q : \quad X = B + s\mathbf{u}, \quad s \in \mathbb{R}
\]

are skewed if and only if the vectors \( \mathbf{u}, \mathbf{v} \), and \( \overrightarrow{AB} \), are not on the same plane, i.e. if and only if the three vectors are linearly independent, thus if and only if the determinant of the matrix formed with the three vectors as rows (or columns) has a non-zero determinant. In our case: \( \mathbf{v} = (5, -2, -3) \), \( \mathbf{u} = (4, -3, -1) \), and \( \overrightarrow{AB} = [-2, 6, 3] - [1, 2, 1] = (-3, 4, 2) \), and the above determinant is:

\[
\det \begin{pmatrix}
5 & -2 & -3 \\
4 & -3 & -1 \\
-3 & 4 & 2
\end{pmatrix} = -21 \neq 0,
\]

thus, the given lines are skewed. In order to find the distance \( d(p, q) \), we may remember the formula expressing this distance as the height of the parallelepiped \( V \) constructed with sides \( \mathbf{u}, \mathbf{v} \) and \( \overrightarrow{AB} \), and with base \( B \), the parallelogram formed by \( \mathbf{v} \) and \( \mathbf{u} \):

\[
d(p, q) = \frac{\text{Volume}(V)}{\text{Area}(B)} = \frac{\det \left( \begin{pmatrix} \mathbf{v} \\ \overrightarrow{AB} \end{pmatrix} \right)}{\|\mathbf{v} \times \mathbf{u}\|} = \frac{|-21|}{\|(-7, -7, -7)\|} = \frac{21}{7\sqrt{3}} = \sqrt{3}.
\]

In case we don’t remember this geometrical fact, we may find directly the points \( P \in p, Q \in q \) such that \( d(p, q) = d(P, Q) \). These points have the property that the vector \( \overrightarrow{PQ} \) is orthogonal to \( \mathbf{v} \) and also to \( \mathbf{u} \). Thus we look for the particular values \( t_0 \) and \( s_0 \) such that \( P = [1, 2, 1] + t_0(5, -2, -3) \), \( Q = [-2, 6, 3] + s_0(4, -3, -1) \), \( \overrightarrow{PQ} = Q - P = [-2, 6, 3] + s_0(4, -3, -1) - [1, 2, 1] + t_0(5, -2, -3) = (-3, 4, 2) + t_0(5, -2, -3) + s_0(4, -3, -1) \) and the following conditions of orthogonality are satisfied:

\[
((-3, 4, 2) + t_0(5, -2, -3) + s_0(4, -3, -1)) \cdot (5, -2, -3) = 0
\]

\[
((-3, 4, 2) + t_0(5, -2, -3) + s_0(4, -3, -1)) \cdot (4, -3, -1) = 0.
\]

This gives the following system:

\[
-15 - 8 - 6 + t_0(25 + 4 + 9) + s_0(20 + 6 + 3) = 0
\]

\[
-12 - 12 - 2 + t_0(20 + 6 + 3) + u_0(16 + 9 + 1) = 0.
\]

After simplification, we get:

\[
38t_0 + 29s_0 = 29
\]

\[
29t_0 + 26s_0 = 26.
\]
that obviously has a unique solution \( t_0 = 0, s_0 = 1 \). Thus \( P = [1, 2, 1] + t_0(5, -2, -3) = [1, 2, 1] = A \), and \( Q = [-2, 6, 3] + s_0(4, -3, -1) = [2, 3, 2] \). From this, we again obtain:

\[
d(p, q) = d(P, Q) = \|PQ\| = \|\begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}\| = \|1, 1, 1\| = \sqrt{3}.
\]

8.21) Given the line \( p \) and two points \( A \) and \( B \), find a point \( M \) on \( p \) that is equally far away from \( A \) and \( B \), where:

\[
p = \rho \cap \sigma, \quad \rho : x + y - z - 3 = 0, \quad \sigma : 3y - 2z + 1 = 0, \quad A = [1, 0, -1], \text{ and } B = [3, 4, 5].
\]

Solution. A general point \( X = [x, y, z] \) of \( p \) has distance from \( A \):

\[
d(A, X) = \|AX\| = \|(x, y, z) - [1, 0, -1]\| = \|(x - 1, y, z + 1)\| = \sqrt{(x - 1)^2 + y^2 + (z + 1)^2},
\]

and distance from \( B \):

\[
d(B, X) = \|BX\| = \|(x, y, z) - [3, 4, 5]\| = \|(x - 3, y - 4, z - 5)\| = \sqrt{(x - 3)^2 + (y - 4)^2 + (z - 5)^2}.
\]

Because \( M \) lies on \( p = \rho \cap \sigma \) and has same distance from \( A \) and \( B \), the point \( M \) that we are looking for must have coordinates that satisfy the following three conditions:

\[
\begin{align*}
x + y - z &= 0, \\
3y - 2z &= 0, \\
\sqrt{(x - 1)^2 + y^2 + (z + 1)^2} &= \sqrt{(x - 3)^2 + (y - 4)^2 + (z - 5)^2}.
\end{align*}
\]

Thus we are looking for the solution of the system:

\[
\begin{align*}
x + y - z &= 0, \\
3y - 2z &= 0, \\
x^2 - 2x + 1 + y^2 + z^2 + 2z + 1 &= x^2 - 6x + 9 + y^2 - 8y + 16 + z^2 - 10z + 25,
\end{align*}
\]

that, after simplification, gives a linear system for which we use a gaussian reduction:

\[
\begin{align*}
x + y - z &= \frac{3}{3y - 2z} = -1, \quad R_3 - R_1 \quad x + y - z &= \frac{3}{3y - 2z} = -1, \quad -3R_3 + R_2 \quad x + y - z &= \frac{3}{3y - 2z} = -1. \\
x + 2y + 3z &= 12, \quad \sim \quad y + 4z &= 9, \quad \sim -14z &= -28.
\end{align*}
\]

Starting from the last equation, we get that the unique solution of the system (the coordinates of the point \( M \)) is \( z = 2, \; y = 1, \; x = 4 \). The point \( M = [4, 1, 2] \) lies on \( p \) and:

\[
d(A, M) = \|(3, 1, 3)\| = d(B, M) = \|(1, -3, -3)\| = \sqrt{19}.
\]

8.23) Find the point \( R \) symmetric to \( P = [-4, 5, 8] \) with respect to the line \( p \) through \( A = [9, 4, 10] \) and \( B = [-6, 1, 1] \).

Solution. We first find the projection \( P' \) of \( P \) on \( p \), and then we look for the point \( R \) such that \( P' \) is the middle point of the segment \( PR \), i.e. \( \frac{P + R}{2} = P' \) or equivalently \( R = 2P' - P \). Looking for \( P' \), we observe that \( P' \) is the point of intersection of the line \( p \) with the plane \( \rho \) going through \( P \) and orthogonal to \( p \). The plane \( \rho \), thus, has orthogonal vector \( \mathbf{n} = \mathbf{v} \), where \( \mathbf{v} = \overrightarrow{AB} \) is the direction vector of \( p \).

\[
\mathbf{v} = \overrightarrow{AB} = B - A = [-6, 1, 1] - [9, 4, 10] = (-15, -3, -9) = -3(5, 1, 3).
\]

We will use for \( p \) direction vector \( \mathbf{v}_1 = (5, 1, 3) \), parallel to \( \mathbf{v} \) but with more ”convenient” coordinates.

\[
p : \quad X = A + t \mathbf{v}, \quad t \in \mathbb{R} \quad \implies \quad X = [9, 4, 10] + t(5, 1, 3).
\]

\[
\rho : \quad (X - P) \cdot \mathbf{v}_1 = 0 \quad \implies \quad (x + 4, y - 5, z - 8) \cdot (5, 1, 3) = 0 \quad \implies \quad 5x + y + 3z - 9 = 0.
\]

To find \( P' \), we look for the particular value \( t_0 \) of \( t \), such that the point of \( p \) with coordinates \([9 + 5t_0, 4 + t_0, 10 + 3t_0]\) satisfies also the equation of \( \rho \):

\[
5(9 + 5t_0) + (4 + t_0) + 3(10 + 3t_0) - 9 = 0.
\]

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After simple computation we get: \[35t_0 + 70 = 0,\]
which means that \(t_0 = -2\). Thus \(P' = [9 + 5(-2), 4 - 2, 10 + 3(-2)] = [-1, 2, 4]\). We can now find \(R\):
\[R = 2P' - P = 2[-1, 2, 4] - [-4, 5, 8] = [2, -1, 0].\]

8.25) A line with direction vector \(\mathbf{v} = (0, 9, -1)\) intersects lines \(p\) and \(q\), find the coordinates of the points of intersection, where
\[p: \quad \frac{x - 8}{5} = \frac{y - 5}{1} = \frac{z}{-1}, \quad \text{and} \quad q: \quad \frac{x}{1} = \frac{y - 1}{-2} = \frac{z + 1}{1}.\]

**Solution.** The given equations for \(p\) and \(q\) are equivalent to:
\[p: \quad X = [8, 5, 0] + t(5, 1, -1), \quad t \in \mathbb{R} \quad \text{and} \quad q: \quad X = [0, 1, -1] + s(1, -2, 1), \quad s \in \mathbb{R}.\]
We are looking for a point \(P \in p\) with coordinates \(P = [8 + 5t, 5 + t, -t]\) for a particular value of \(t\), and a point \(Q \in q\), \(Q = [s, 1 - 2s, -1 + s]\), for a particular value of \(s\), such that the vector \(\overrightarrow{PQ}\) is parallel to the given \(\mathbf{v}\), i.e. \(\overrightarrow{PQ} = \lambda \mathbf{v}\), for a particular value of \(\lambda\). We get the following condition:
\[[s, 1 - 2s, -1 + s] - [8 + 5t, 5 + t, -t] = \lambda(0, 9, -1),\]
that is equivalent to a system of linear equations:
\[
\begin{align*}
-8 - 5t &= 0, \\
1 - 2s - 5t &= 9\lambda, \\
-1 + s + t &= -\lambda.
\end{align*}
\]
After reordering, we start a gaussian reduction:
\[
\begin{align*}
s - 5t &= 8, \\
9\lambda + 2s + t &= -4, \\
\lambda + s + t &= 1
\end{align*}
\]
\[
\begin{align*}
R_3 &\leftrightarrow R_1, \\
R_3 &\leftrightarrow R_2, \\
R_2 - 8R_3 &= 8, \\
R_2 + 7R_3 &= 43.
\end{align*}
\]
Starting from the last equation, we get that the unique solution of the system is \(t = -1, \ s = 3, \ \lambda = -1\). Thus, the sought points are \(P = [8 + 5(-1), 5 + (-1), -(-1)] = [3, 4, 1]\), and \(Q = [3, 1 - 2(3), -1 + 3] = [3, -5, 2]\).

8.27) Find the distance of the point \(A = [3, 2, 1]\) from the plane containing the lines \(p\) and \(q\), where
\[p: \quad \frac{x + 1}{1} = \frac{y - 3}{2} = \frac{z - 2}{2}, \quad \text{and} \quad q: \quad X = [0, 4, 2] + t(1, 1, 0), \quad t \in \mathbb{R}.\]

**Solution.** First of all, we want to check that the given lines \(p\) and \(q\) really lie on the same plane and they are not skewed. Writing the equation of \(p\) in vector form: \(p: \quad X = [-1, 3, 2] + s(1, 2, 2), \quad s \in \mathbb{R}\), we need to verify that the vectors \(\mathbf{v}_p = (1, 2, 2), \quad \mathbf{v}_q = (1, 1, 0)\) and \(\mathbf{w} = ([1, 3, 2] - [0, 4, 2])\), are linearly dependent. Since \(\mathbf{w} = (-1, -1, 0)\), the vectors are obviously linearly dependent, thus \(p\) and \(q\) lie on the same plane \(\sigma\). In Solution of Example 7, we discussed two ways how to solve the general problem: find the distance of a point \(A\) from a plane \(\sigma\). Using the first method: we need to identify a point \(Q\) on \(\sigma\) and a vector \(\mathbf{n}\), orthogonal to \(\sigma\). We can choose as \(Q\) the "starting" point of line \(q\), \(Q = [0, 4, 2]\). To find \(\mathbf{n}\) we will consider the direction vectors \(\mathbf{v}_p\), \(\mathbf{v}_q\) of \(p\) and \(q\), and set \(\mathbf{n} = \mathbf{v}_p \times \mathbf{v}_q\). Since \(\mathbf{v}_p = (1, 2, 2)\) and \(\mathbf{v}_q = (1, 1, 0)\), we have:
\[
\mathbf{n} = \mathbf{v}_p \times \mathbf{v}_q = \det \begin{pmatrix} i & j & k \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix} = (-2, 2, -1).
\]

Now, we may conclude:
\[
d(A, \sigma) = \frac{\|\mathbf{AQ}\| \cdot \|\mathbf{n}\|}{\|\mathbf{n}\|} = \frac{\|\mathbf{[0, 4, 2] - [3, 2, 1]}\cdot (-2, 2, -1)\|}{\|(-2, 2, 1)\|} = \frac{\|(-3, 2, 1)\cdot (-2, 2, -1)\|}{\|(-2, 2, -1)\|} = \frac{6 + 4 - 1}{\sqrt{4 + 4 + 1}} = 3.
\]

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9.1) Consider the following system of linear equations

\[
(S) \begin{cases} 
  kx + y - z &= -k \\
  (1 - k)y + z &= h + k \\
  y + (1 - k)z &= 2h + 1
\end{cases}
\]

where \( h, k \in \mathbb{R} \) are real parameters.

9.1.a) Determine for what values of \( h, k \in \mathbb{R} \) the system has one unique solution, no solutions or infinitely many solutions.

9.1.b) If \( A_k \) is the matrix associated to the system \( (S) \), so that the system can be written in the form

\[
A_k \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -k \\ h + k \\ 2h + 1 \end{pmatrix}
\]

write the linear transformation \( l_k : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) associated to \( A_k \) with respect to the canonical basis of \( \mathbb{R}^3 \).

9.1.c) Determine for what values of \( k \) \( \dim \text{Im} l_k = 3 \).

9.1.d) Determine for what values of \( k \) \( \dim \ker l_k = 2 \).

9.1.e) For \( k = 0 \) is the transformation \( l_0 \) diagonalizable?

9.2) Given the linear transformations \( f : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \), and \( g : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \),

\[
f(x, y, z, t) = (x - t, x + y, z + y), \quad g(x, y, z) = (z - x, y, y, x + t).
\]

9.2.a) Find \( \ker(f) \), \( \text{Im}(f) \), \( \ker(g) \), \( \text{Im}(g) \).

9.2.b) Solve \( g(x, y, z) = (h, -1, h, 1) \) for any possible \( h \in \mathbb{R} \).

9.2.c) Write down the general form of \( f \circ g \), \( g \circ f \) and determine if they are isomorphisms.

9.2.d) Are \( f \circ g \), \( g \circ f \) diagonalizable? Find all respective eigenvalues and a basis for each corresponding eigenspace.

9.3) Consider the linear transformation \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by:

\[
f(1, 0, 0) = (-1, -2, -3) \\
f(0, 1, 0) = (2, 2, 2) \\
f(0, 0, 1) = (0, 1, 2)
\]

9.3.a) Is \( f \) invertible?

9.3.b) Find a basis of \( \ker(f) \) and a basis of \( \text{Im}(f) \).

9.3.c) Find all eigenvalues and corresponding eigenvectors of \( f \). Is \( f \) diagonalizable?

9.3.d) Solve the system \( f^2(v) = 0 \).

9.4) Given \( f_h : \mathbb{P}^3(x) \rightarrow \mathcal{M}^{2,2} \) defined by

\[
f_h(a + bx + cx^2 + dx^3) = \begin{pmatrix} a + b + c + hd & b + c \\ -b - c - hd & hb \end{pmatrix},
\]

where \( h \in \mathbb{R} \) is a parameter.

9.4.a) Give the definition of Kernel and Image of a linear transformation. Find, for all possible values of \( h \), \( \ker(f_h) \), \( \text{Im}(f_h) \), their bases and dimensions.

9.4.b) Does there exist a value of \( h \in \mathbb{R} \) such that \( f_h \) is not an isomorphism?
9.4.c) Determine \( f_h^{-1}(A) \) (i.e. the set of all polynomials mapped to \( A \)), where \( A = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \).

9.4.d) For \( h = 0 \), find all eigenvalues and corresponding eigenvectors of \( f_0 \) determine if \( f_0 \) is diagonalizable.

9.5) Given the transformation \( f_h : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by

\[
f_h(x, y, z) = (x - hz, x + y - hz, -hx + z),
\]

where \( h \in \mathbb{R} \) is a parameter.

9.5.a) Find, for all possible values of \( h \), \( \text{Ker}(f_h) \), \( \text{Im}(f_h) \), their bases and dimensions.

9.5.b) Is \( f_h \) an isomorphism for some value of \( h \)?

9.5.c) Determine \( f_h^{-1}(1, 0, 1) = \{ (x, y, z) \in \mathbb{R}^3 : f_h(x, y, z) = (1, 0, 1) \} \).

9.5.d) Give the definition of eigenvalue and eigenvector of a linear transformation. Determine if \( f_h \) is diagonalizable for some values of \( h \). For \( h = 1 \), find a basis of \( \mathbb{R}^3 \) made of eigenvectors of \( f_1 \).

9.6) Consider the matrix \( A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -3 & h \\ 2 & -h & 1 \end{pmatrix} \).

9.6.a) Give the definition of regular matrix. Find all values of \( h \) for which the matrix \( A \) is regular.

9.6.b) For \( h = 1 \), solve the matrix equation \( A \cdot X = B \),

where \( B = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \).

9.6.c) Find the solutions of the system \( A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 12 \end{pmatrix} \) for all possible \( h \in \mathbb{R} \).

9.6.d) Is the matrix \( A \) diagonalizable for \( h = 0 \)?

9.6.e) For \( h = 3 \), find the Image and the Kernel of the linear transformation \( l \) that has \( A \) as associated matrix (with respect to the standard basis of \( \mathbb{R}^3 \)).
9.1) Consider the following system of linear equations

\[
\begin{align*}
S : \begin{cases}
kx + y - z &= -k \\
(1-k)y + z &= h + k \\
y + (1-k)z &= 2h + 1
\end{cases}
\end{align*}
\]

where \( h, k \in \mathbb{R} \) are real parameters.

9.1.a) Determine for what values of \( h, k \in \mathbb{R} \) the system has one unique solution, no solutions or infinitely many solutions.

9.1.b) If \( A_k \) is the matrix associated to the system \( S \), so that the system can be written in the form

\[
A_k \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -k \\ h + k \\ 2h + 1 \end{pmatrix}
\]

write the linear transformation \( l_k : \mathbb{R}^3 \to \mathbb{R}^3 \) associated to \( A_k \) with respect to the canonical basis of \( \mathbb{R}^3 \).

9.1.c) Determine for what values of \( k \) \( \dim \text{Im} l_k = 3 \).

9.1.d) Determine for what values of \( k \) \( \dim \text{Ker} l_k = 2 \).

9.1.e) For \( k = 0 \) is the transformation \( l_0 \) diagonalizable?

Solution. 9.1.a) Let’s consider the matrix \( A_k \) associated to the homogeneous system:

\[
A_k = \begin{pmatrix} k & 1 & -1 \\ 0 & 1-k & 1 \\ 0 & 1 & 1-k \end{pmatrix}.
\]

Evaluating \( \det A_k = k^2(k-2) \), we see that, for \( k \neq 0 \) and \( k \neq 2 \) (and any possible value of \( h \)), the system has a unique solution that can be evaluated using Cramer’s rule.

For \( k = 0 \), the augmented matrix of the system is

\[
A_0 = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & h \\ 0 & 1 & 1 & 2h \end{pmatrix}.
\]

Let’s use a matrix form of Gaussian reduction: with operations \( R_2 - R_1 \), and \( R_3 - R_2 \) we get:

\[
\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & h \\ 0 & 0 & 0 & h \end{pmatrix}.
\]

Now, using Frobenius theorem, we see that, for \( k = 0 \) and \( h \neq 0 \), the system has no solution since the rank of the augmented matrix is not equal to the rank of the matrix associated to the homogeneous system; while for \( k = 0 \) and \( h = 0 \) the system has infinitely many solution since the rank of the augmented matrix (equal to the rank of the matrix associated to the homogeneous system) is two and the number of unknowns is three.

In a similar, for \( k = 2 \), the augmented matrix of the system is

\[
A_2 = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 0 & -1 & 1 & h + 2 \\ 0 & 1 & -1 & 2h + 1 \end{pmatrix}.
\]

We start a matrix form of Gaussian reduction with operations \( R_3 + R_2 \) and we get:

\[
\begin{pmatrix} 2 & 1 & -1 & -2 \\ 0 & -1 & 1 & h + 2 \\ 0 & 0 & 0 & 3h + 3 \end{pmatrix}.
\]

Again, using Frobenius theorem, we see that, for \( k = 2 \) and \( h \neq -1 \), the system has no solution since the rank of the augmented matrix is not equal to the rank of the matrix associated to the homogeneous system;
while for \( k = -2 \) and \( h = -1 \) the system has infinitely many solution since the rank of the augmented matrix (equal to the rank of the matrix associated to the homogeneous system) is two and the number of unknowns is three.

9.1.b) We have seen that

\[
A_k = \begin{pmatrix}
  k & 1 & -1 \\
  0 & 1 - k & 1 \\
  0 & 1 & 1 - k \\
\end{pmatrix}.
\]

To find the linear transformation \( l_k : \mathbb{R}^3 \to \mathbb{R}^3 \) associated to \( A_k \) with respect to the canonical basis of \( \mathbb{R}^3 \), we just recall that, for every \( (x, y, z) \in \mathbb{R}^3 \),

\[
l_k(x, y, z) = A_k \begin{pmatrix}
  x \\
  y \\
  z \\
\end{pmatrix} = (kx + y - z, (1-k)y + z, y + (1-k)z).
\]

9.1.c) We now need to find the values of \( k \) for which \( \dim \text{Im} l_k = 3 \). The theorem on dimensions of Kernel and Image of a linear transformation \( l_k \) from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) states that

\[
3 = \dim \mathbb{R}^3 = \dim \ker l_k + \dim \text{im} l_k,
\]

thus the Image of \( l_k \) has dimension three if and only if \( l_k \) is an isomorphism. This happens for all values of \( k \) for which \( \det A_k \neq 0 \), thus \( \dim \text{im} l_k = 3 \) for all \( k \neq 0, -2 \).

9.1.d) We now determine for what values of \( k \) dim Ker \( l_k = 2 \). We have seen that for \( k \neq 0, -2 \), \( l_k \) is an isomorphism, moreover in 9.1.a) we have seen that for \( k = 0 \) and \( k = -2 \) the rank of \( A_k \) is two, this implies that the dimension of the kernel is one (since the dimension of the Kernel is equal to the dimension of the space of solutions of the homogeneous system). We conclude that there is no value of \( k \) for which \( \dim \ker l_k = 2 \).

9.1.e) For \( k = 0 \), the transformation \( l_0(x, y, z) = (y - z, y + z, y + z) \) has associated matrix

\[
A_0 = \begin{pmatrix}
  0 & 1 & -1 \\
  0 & 1 & 1 \\
  0 & 1 & 1 \\
\end{pmatrix}.
\]

To see if \( A_0 \) is diagonalizable, we start by finding the real solutions of \( \det(A_0 - \lambda I) = 0 \). In this case, we have:

\[
\det \left( \begin{pmatrix}
  0 & 1 & -1 \\
  0 & 1 & 1 \\
  0 & 1 & 1 \\
\end{pmatrix} - \lambda \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 \\
\end{pmatrix} \right) = \det \begin{pmatrix}
  -\lambda & 1 & -1 \\
  0 & 1 - \lambda & 1 \\
  0 & 1 & 1 - \lambda \\
\end{pmatrix} = \lambda^2(2 - \lambda).
\]

The characteristic equation of \( A_0 \) has two distinct solutions \( \lambda = 2 \) with algebraic multiplicity one and \( \lambda = 0 \) with algebraic multiplicity two. We now look for the eigenvectors corresponding to \( \lambda = 0 \). We must find the nontrivial solutions of the homogeneous system \((A_0 - \lambda I)\mathbf{v} = \mathbf{0}\), where \( \lambda = 0 \). The augmented matrix of the system is

\[
\begin{pmatrix}
  0 & 1 & -1 & 0 \\
  0 & 1 & 1 & 0 \\
  0 & 1 & 1 & 0 \\
\end{pmatrix} \sim \begin{pmatrix}
  R_2 - R_1 & \quad 0 & 1 & -1 & 0 \\
  R_2 - R_1 & \quad 0 & 0 & 2 & 0 \\
  R_2 - R_1 & \quad 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

All the eigenvectors corresponding to \( \lambda = 0 \) are \( \{(1, 0, 0) : t \in \mathbb{R} \setminus \{0\}\} \), and the eigenspace corresponding to \( \lambda = 0 \) is \( U_0 = \text{span}\{(1, 0, 0)\} \). Corresponding to the eigenvalue \( \lambda = 0 \) with algebraic multiplicity two, we found and eigenspace \( U_0 \) with dimension one. This implies that \( A_0 \), and thus \( l_0 \), is not diagonalizable.

9.3) Consider the linear transformation \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by:

\[
f(1,0,0) = (-1,-2,-3) \\
f(0,1,0) = (2,2,2) \\
f(0,0,1) = (0,1,2)
\]

9.3.a) Is \( f \) invertible?

9.3.b) Find a basis of Ker\( (f) \) and a basis of Im\( (f) \).

9.3.c) Find all eigenvalues and corresponding eigenvectors of \( f \). Is \( f \) diagonalizable?

9.3.d) Solve the system \( f^2(v) = 0 \).
Solution. We consider the ordered canonical basis of \( \mathbb{R}^3 \), \( C = \{(1,0,0), (0,1,0), (0,0,1)\} \), and we will find the matrix \( A = A(l, C, C) \) associated to the given linear transformation with respect to \( C \). We are given that:

\[
\begin{align*}
f(1,0,0) &= (-1,-2,-3) \\
f(0,1,0) &= (2,2,2) \\
f(0,0,1) &= (0,1,2)
\end{align*}
\]

Thus we can write \( A = \begin{pmatrix} -1 & 2 & 0 \\ -2 & 2 & 1 \\ -3 & 2 & 2 \end{pmatrix} \).

9.3.a) We evaluate \( \det A \), because, only in case \( \det A \neq 0 \), we know that the linear transformation is an isomorphism thus invertible. Since \( \det A = -4 - 6 + 2 + 8 = 0 \), we conclude that \( f \) is not invertible.

9.3.b) We now find \( \ker f = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} \), the space of solutions of the homogeneous system associated to \( A \). Since \( \det A = 0 \), we know that the homogeneous system does not have a unique solution (the trivial one). We will use the matrix form of Gaussian reduction and get:

\[
\begin{pmatrix} -1 & 2 & 0 \\ -2 & 2 & 1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} R_2 - 2R_1 \\ -2R_2 \\ R_3 - 2R_2 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Setting \( x_2 = \alpha, x_3 = 2\alpha \), and subsequently, from the first row \( x_1 = \alpha \). Thus we conclude that:

\[
\ker f = \{ (2\alpha, \alpha, 2\alpha) : \alpha \in \mathbb{R} \} = \text{span}\{ (2,1,2) \},
\]

A basis of \( \ker f \) is \( B = \{ (2,1,2) \} \).

From the Theorem on dimensions, we get \( \dim \text{Im} l = \dim \mathbb{R}^3 - \dim \ker f = 2 \), and, knowing that

\[
\text{Im} l = \text{span}\{ l(1,0,0), l(0,1,0), l(0,0,1) \}
\]

we get

\[
\text{Im} l = \text{span}\{ (-1,-2,-3), (2,2,2), (0,1,2) \} = \text{span}\{ (-1,-2,-3), (2,2,2) \},
\]

where the first two vectors were chosen randomly, just considering that, because they are linearly independent (they are not a multiple of each other), they must span the space \( \text{Im} l \) that has dimension 2. Thus, a basis of \( \text{Im} l \) is \( B_l = \{ (-1,-2,-3), (2,2,2) \} \).

9.3.c) To find all eigenvalues and corresponding eigenvectors of \( f \) (or equivalently of \( A \)), we start by finding the real solutions of \( \det(A - \lambda I) = 0 \). In this case, we have:

\[
\det \begin{pmatrix} -1 & 2 & 0 \\ -2 & 2 & 1 \\ -3 & 2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \det \begin{pmatrix} -1 - \lambda & 2 & 0 \\ -2 & 2 - \lambda & 1 \\ -3 & 2 & -\lambda \end{pmatrix} = -\lambda(\lambda - 1)(\lambda - 2).
\]

The characteristic equation of \( A \) has three distinct solutions with algebraic multiplicity one: \( \lambda = 0, \lambda = 1 \) and \( \lambda = 2 \) are the eigenvalues of \( A \). We may already conclude that \( A \) (thus \( f \)) is diagonalizable.

We now look for the eigenvectors corresponding to \( \lambda = 0 \). We must find the nontrivial solutions of the homogeneous system \((A - \lambda I)v = 0\), where \( \lambda = 0 \). The augmented matrix of the system is

\[
\begin{pmatrix} -1 & 2 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ -3 & 2 & 2 & 0 \end{pmatrix}.
\]

We don’t need to evaluate anything, since we have already done the math looking for the Kernel of \( f \). All the eigenvectors corresponding to \( \lambda = 0 \) are \( \{ t(2,1,2) : t \in \mathbb{R} \setminus \{0\} \} \), and the eigenspace corresponding to \( \lambda = 0 \) is (the kernel of \( f \)) \( U_0 = \text{span}\{ (2,1,2) \} \).

Corresponding to the eigenvalue \( \lambda = 1 \), we look for eigenvectors, i.e. the nontrivial solutions of the homogeneous system \((A - \lambda I)v = 0\), where \( \lambda = 1 \). The augmented matrix of the system is

\[
\begin{pmatrix} -2 & 2 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} R_1/2 \\ R_2 - R_1 \\ 2R_3 - 3R_1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} R_3 - 2R_2 \\ R_3 - 2R_2 \\ 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

All the eigenvectors corresponding to \( \lambda = 1 \) are \( \{ t(1,1,1) : t \in \mathbb{R} \setminus \{0\} \} \), and the eigenspace corresponding to \( \lambda = 1 \) is \( U_1 = \text{span}\{ (1,1,1) \} \).

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Finally, we look for the eigenvectors corresponding to the eigenvalue \( \lambda = 2 \), i.e. the look for the nontrivial solutions of the homogeneous system \((A - \lambda I)\mathbf{v} = \mathbf{0}\), where \( \lambda = 2 \). The augmented matrix of the system is
\[
\begin{pmatrix}
-3 & 2 & 0 & 0 \\
-2 & 0 & 1 & 0 \\
-3 & 2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
R_3 - R_1 \\
\sim
\end{pmatrix}
\begin{pmatrix}
-3 & 2 & 0 & 0 \\
-2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
All the eigenvectors corresponding to \( \lambda = 2 \) are \( \{t(2, 3, 4) : t \in \mathbb{R} \setminus \{0\}\} \), and the eigenspace corresponding to \( \lambda = 2 \) is \( U_2 = \text{span}\{2, 3, 4\} \).

9.3.d) To solve the system \( f^2(\mathbf{v}) = \mathbf{0} \), is equivalent to finding the Kernel of \( f^2 \). Indeed, we recall that, given a linear transformation \( f \), with associated matrix \( A \), the transformation \( f^2 = f \circ f \) is again linear and has associated matrix \( A^2 = A \cdot A \). Let’s evaluate \( A^2 \):
\[
A^2 = A \cdot A = \begin{pmatrix}
-1 & 2 & 0 \\
-2 & 2 & 1 \\
-3 & 2 & 2
\end{pmatrix}
\begin{pmatrix}
-1 & 2 & 0 \\
-2 & 2 & 1 \\
-3 & 2 & 2
\end{pmatrix} = \begin{pmatrix}
-3 & 2 & 2 \\
-5 & 2 & 4 \\
-7 & 2 & 6
\end{pmatrix}.
\]
We know that \( \text{rank} A^2 \leq \text{rank} A \) (and also \( \text{Ker} f \subseteq \text{Ker} f^2 \)). Using a Gaussian reduction, we evaluate:
\[
\begin{pmatrix}
-3 & 2 & 2 \\
-5 & 2 & 4 \\
-7 & 2 & 6
\end{pmatrix}
\begin{pmatrix}
3R_2 - 5R_1 \\
-3 & 2 & 2 \\
0 & -8 & 4
\end{pmatrix}
\sim
\begin{pmatrix}
3R_2 - 5R_1 \\
0 & -8 & 4 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
R_2/2 \\
R_3 - 2R_2 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
-3 & 2 & 2 \\
0 & -2 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]
From this, it follows that \( \text{rank} A^2 = \text{rank} A \), and thus \( \text{Ker} f^2 = \text{Ker} f = \text{span}\{(2, 1, 2)\} \).

9.5) Given the transformation \( f_h : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by
\[
f_h(x, y, z) = (x - hz, x + y - hz, -hx + z),
\]
where \( h \in \mathbb{R} \) is a parameter.

9.5.a) Find, for all possible values of \( h \), \( \text{Ker}(f_h) \), \( \text{Im}(f_h) \), their bases and dimensions.

9.5.b) Is \( f_h \) an isomorphism for some value of \( h \)?

9.5.c) Determine \( f_h^{-1}(1, 0, 1) = \{(x, y, z) \in \mathbb{R}^3 : f_h(x, y, z) = (1, 0, 1)\} \).

9.5.d) Give the definition of eigenvalue and eigenvector of a linear transformation. Determine if \( f_h \) is diagonalizable for some values of \( h \). For \( h = 1 \), find a basis of \( \mathbb{R}^3 \) made of eigenvectors of \( f_1 \).

**Solution.** We consider the ordered canonical basis of \( \mathbb{R}^3 \), \( C = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \), and we will find the matrix \( A_h = A_h(f_h, C, C) \) associated to the given linear transformation with respect to \( C \). We evaluate:
\[
f_h(1, 0, 0) = (1, 1, -h) \\
f_h(0, 1, 0) = (0, 1, 0) \\
f_h(0, 0, 1) = (-h, -h, 1)
\]
Thus, we can write
\[
A_h = \begin{pmatrix}
1 & 0 & -h \\
1 & 1 & -h \\
0 & 0 & 1
\end{pmatrix}.
\]

9.5.a) To find \( \text{Ker} f_h = \{(x, y, z) \in \mathbb{R}^3 : A_h \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}\} \), the space of solutions of the homogeneous system associated to \( A_h \), we first evaluate \( \det A_h \), because, for all values of \( h \in \mathbb{R} \) such that \( \det A_h \neq 0 \), we know that the homogeneous system has a unique solution, the trivial one, thus \( \text{Ker} f_h = \{0\} \).

Since \( \det A_h = 1 - h^2 = (1 - h)(1 + h) \), we conclude that, for \( h \neq 1, h \neq -1 \), \( \text{Ker} f_h = \{0, 0, 0\} \), consequently \( \text{Im} f_h = \mathbb{R}^3 \), that means that \( f_h \) is an isomorphism (9.5.b)).

For \( h = 1 \), we need to find a basis of
\[
\text{Ker} f_1 = \{(x, y, z) \in \mathbb{R}^3 : A_1 \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}\} = \{(x, y, z) \in \mathbb{R}^3 : \begin{pmatrix}
1 & 0 & -1 \\
1 & 1 & -1 \\
-1 & 0 & 1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}\},
\]

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that means, we need to solve the homogeneous system:

\[
\begin{pmatrix}
1 & 0 & -1 & 1 \\
1 & 1 & -1 & 0 \\
-1 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
R_2 - R_1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Setting \( z = \alpha, \alpha \in \mathbb{R} \), we get from the first row \( x = \alpha \), while \( y = 0 \) follows from the second row. Thus we conclude that:

\( \ker f_1 = \{ (\alpha, 0, \alpha) \mid \alpha \in \mathbb{R} \} = \text{span}\{(1, 0, 1)\} \), \( \dim \ker f_1 = 1 \), and a basis of \( \ker f_1 \) is \( B_{K_1} = \{ (1, 0, 1) \} \).

From the Theorem on dimensions, we get \( \dim \text{im} f_1 = \dim \mathbb{R}^3 - \dim \ker f_1 = 2 \), moreover:

\( \text{im} f_1 = \text{span}\{f_1(1, 0, 0), f_1(0, 1, 0), f_1(0, 0, 1)\} = \text{span}\{(1, 1, -1), (0, 1, 0), (-1, -1, 1)\} = \text{span}\{(1, 1, -1), (0, 1, 0)\} \).

\( \text{im} f_1 \) has dimension 2, and a basis of \( \text{im} f_1 \) is \( B_{I_1} = \{ (1, 1, -1), (0, 1, 0) \} \).

For \( h = -1 \), we need to find a basis of

\[
\ker f_{-1} = \left\{ (x, y, z) \in \mathbb{R}^3 : A_{-1}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \left\{ (x, y, z) \in \mathbb{R}^3 : A_{-1}
\begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\},
\]

that means, we need to solve the homogeneous system:

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
R_2 - R_1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Setting \( z = \alpha, \alpha \in \mathbb{R} \), we get from the first row \( x = -\alpha \), while \( y = 0 \) follows from the second row. Thus we conclude that:

\( \ker f_{-1} = \{ (\alpha, 0, -\alpha) \mid \alpha \in \mathbb{R} \} = \text{span}\{(1, 0, -1)\} \), \( \dim \ker f_{-1} = 1 \), and a basis of \( \ker f_{-1} \) is \( B_{K_{-1}} = \{ (1, 0, -1) \} \).

From the Theorem on dimensions, we get \( \dim \text{im} f_{-1} = \dim \mathbb{R}^3 - \dim \ker f_{-1} = 2 \), moreover:

\( \text{im} f_{-1} = \text{span}\{f_{-1}(1, 0, 0), f_{-1}(0, 1, 0), f_{-1}(0, 0, 1)\} = \text{span}\{(1, 1, 1), (0, 1, 0), (1, 1, 1)\} = \text{span}\{(1, 1, 1), (0, 1, 0)\} \).

\( \text{im} f_{-1} \) has dimension 2, and a basis of \( \text{im} f_{-1} \) is \( B_{I_{-1}} = \{ (1, 1, 1), (0, 1, 0) \} \).

9.5.c) Now we determine \( f_h^{-1}(1, 0, 1) = \{ (x, y, z) \in \mathbb{R}^3 : f_h(x, y, z) = (1, 0, 1) \} \), for any possible value of \( h \in \mathbb{R} \).

For \( h \neq 1, h \neq -1 \), we have seen that \( f_h \) is an isomorphism, thus the system \( f_h(x, y, z) = (1, 0, 1) \) has a unique solution. The augmented matrix of the system is

\[
\begin{pmatrix}
1 & 0 & -h & 1 \\
1 & 1 & -h & 0 \\
-h & 0 & 1 & 1
\end{pmatrix}.
\]

We may evaluate its unique solution (for \( h \neq 1, h \neq -1 \)) using Cramer’s rule:

\[
x = \frac{\det C_1}{\det A_h} = \frac{\det
\begin{pmatrix}
1 & 0 & -h \\
0 & 1 & -h \\
1 & 0 & 1
\end{pmatrix}}{(1-h)(1+h)} = \frac{1 + h}{(1-h)(1+h)} = \frac{1}{1-h},
\]

\[
y = \frac{\det C_2}{\det A_h} = \frac{\det
\begin{pmatrix}
1 & 1 & -h \\
1 & 0 & -h \\
-h & 1 & 1
\end{pmatrix}}{(1-h)(1+h)} = \frac{h^2 - 1}{(1-h)(1+h)} = -1,
\]

\[
z = \frac{\det C_3}{\det A_h} = \frac{\det
\begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
-h & 0 & 1
\end{pmatrix}}{(1-h)(1+h)} = \frac{1 + h}{(1-h)(1+h)} = \frac{1}{1-h}.
\]
For $h = 1$, we will perform a Gaussian reduction starting from the augmented matrix of the given system, where we substitute the value $h = 1$:

$$
\begin{pmatrix}
1 & 0 & -1 & 1 \\
1 & 1 & -1 & 0 \\
-1 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
R_2 - R_1 \\
R_3 + R_1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 2
\end{pmatrix}.
$$

From the last row, we see that the system has no solution.

For $h = -1$, we will perform a Gaussian reduction starting from the augmented matrix of the given system, where we substitute the value $h = -1$:

$$
\begin{pmatrix}
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
R_2 - R_1 \\
R_3 - R_1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

The system has infinitely many solutions. We have already evaluated that $\text{Ker } f = \text{span}\{1, 0, -1\}$, guessing a particular solution of the system $x_p = (0, -1, 0)$, we conclude that all solutions of the system are:

$$x = (0, -1, 0) + \text{span}\{(1, 0, -1)\}.$$

9.5.d) By definition, given a linear transformation $t : M \to L$ ($M, L$ linear spaces), a real number $\lambda \in \mathbb{R}$ is called an eigenvalue of $t$ if there exists a nontrivial vector $v \in M$ ($v \neq 0$), such that $tv = \lambda v$. The vector $v$ is said to be an eigenvector of $t$, associated to the eigenvalue $\lambda$. The set of all eigenvectors associated to the same eigenvalue $\lambda$, plus the zero vector, form a linear subspace of $M$, called eigenspace associated to the eigenvalue $\lambda$.

The transformation $f_h(x, y, z) = (x - hx, x + y - hz, -hx + z)$, has associated matrix

$$A_h = \begin{pmatrix}
1 & 0 & -h \\
1 & 1 & -h \\
-h & 0 & 1
\end{pmatrix}.$$

To see if $f_h$ is diagonalizable, we start by finding the real solutions of $\det(A_h - \lambda I) = 0$. In this case, we have:

$$\det\left(\begin{pmatrix}
1 & 0 & -h \\
1 & 1 & -h \\
-h & 0 & 1
\end{pmatrix} - \lambda \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}\right) = \det\begin{pmatrix}
1 - \lambda & 0 & -h \\
1 & 1 - \lambda & -h \\
-h & 0 & 1 - \lambda
\end{pmatrix} = (1 - \lambda)^3 - (1 - \lambda)h^2 = (1 - \lambda)((1 - \lambda)^2 - h^2) = (1 - \lambda)(1 - \lambda - h)(1 - \lambda + h).
$$

The solutions of the characteristic equation are $\lambda = 1$, $\lambda = 1 - h$, and $\lambda = 1 + h$, thus for any $h \neq 0$, the found eigenvalues are distinct, each with algebraic multiplicity one. It follows that the corresponding $f_h$ is diagonalizable.

For $h = 0$, we must determine if $A_0 = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}$ is diagonalizable. We have calculated the eigenvalues of $A_0$: we found that $\lambda = 1$ is the only eigenvalue with algebraic multiplicity three. Let’s now evaluate the eigenvectors corresponding to $\lambda = 1$. We must find the nontrivial solutions of the homogeneous system $(A_0 - \lambda I)v = 0$, where $\lambda = 1$. The augmented matrix of the system is

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$ 

All eigenvalues of $A_0$ are \{$t(0, 1, 0) + u(0, 0, 1) : t, u \in \mathbb{R}$ non both zero$\},$ and the eigenspace corresponding to $\lambda = 1$ is $U_1 = \text{span}\{(0, 1, 0), (0, 0, 1)\}$. Since the eigenvalue $\lambda = 1$ has algebraic multiplicity three, but the corresponding eigenspace has dimension two, $A_0$, and thus $f_0$, is not diagonalizable.

We conclude that $f_h$ is diagonalizable for any possible real value of $h \neq 0$.

Let’s now consider the value $h = 1$ and find the eigenvectors of $f_1$. In this case the eigenvalues are $\lambda = 1$, $\lambda = 0$, and $\lambda = 2$.

For $\lambda = 1$, we must find the nontrivial solutions of the homogeneous system $(A_1 - \lambda I)v = 0$, where $\lambda = 1$. The augmented matrix of the system is

$$\begin{pmatrix}
0 & 0 & -1 & 0 \\
1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.$$
All the eigenvectors corresponding to $\lambda = 1$ are \{$(t(0, 1, 0) : t \in \mathbb{R} \setminus \{0\}\}$, and the eigenspace corresponding to $\lambda = 1$ is $U_1 = \text{span}\{(0, 1, 0)\}$.

For $\lambda = 0$, we must find the nontrivial solutions of the homogeneous system $(A_1 - \lambda I)v = 0$, where $\lambda = 0$. The augmented matrix of the system is

$$
\begin{pmatrix}
1 & 0 & -1 & 0 \\
1 & 1 & -1 & 0 \\
-1 & 0 & 1 & 0 \\
\end{pmatrix}.
$$

All the eigenvectors corresponding to $\lambda = 0$ are \{$(t(1, 0, 1) : t \in \mathbb{R} \setminus \{0\}\}$, and the eigenspace corresponding to $\lambda = 0$ is $U_0 = \text{span}\{(1, 0, 1)\}$.

For $\lambda = 2$, we must find the nontrivial solutions of the homogeneous system $(A_1 - \lambda I)v = 0$, where $\lambda = 2$. The augmented matrix of the system is

$$
\begin{pmatrix}
-1 & 0 & -1 & 0 \\
1 & -1 & -1 & 0 \\
-1 & 0 & -1 & 0 \\
\end{pmatrix}.
$$

All the eigenvectors corresponding to $\lambda = 2$ are \{$(t(1, 2, -1) : t \in \mathbb{R} \setminus \{0\}\}$, and the eigenspace corresponding to $\lambda = 2$ is $U_2 = \text{span}\{(1, 2, -1)\}$.

For $h = 1$, we found a basis $B$ of $\mathbb{R}^3$ made of eigenvectors of $f_1$:

$$B = \{(0, 1, 0), (1, 0, 1), (1, 2, -1)\}.$$