Associativity and distributivity-like properties in generalized effect algebras

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Abstract

We prove "large" associativity of the partial plus operation in generalized effect algebras and present an overview of distributivity-like properties of partial operations plus and minus in generalized effect algebras with respect to (possibly infinite) suprema and infima and vice versa. These results generalize previous results in various subclasses of generalized effect algebras.

Keywords: Generalized effect algebra; partial operations; supremum; infimum; associativity; distributivity.

1 Introduction

The concept of an effect algebra (under a different name and with a different set of axioms) appeared in the eighties of the twentieth century [6, 7] as an effort to axiomatize the structure of effects in a quantum-mechanical system. The notion of an effect algebra (with a simplified set of axioms) was introduced in [5]. An equivalent notion of a D-poset (using axioms for the partial difference operation) was introduced in [9].

Effect algebras are "unsharp" generalizations of "sharp" quantum logics (orthomodular lattices, orthomodular posets, orthoalgebras) incorporating some fuzzy logics (MV-algebras). E.g., consider the effect algebra $([0,1],\oplus,0,1)$ with the real unit interval $[0,1]$ and the partial operation $\oplus$ defined as the sum of real numbers whenever this sum belongs to $[0,1]$. This effect algebra corresponds to MV-algebra with the Łukasiewicz t-conorm $\oplus$ if we extend the definition of $\oplus$ by $a\oplus b=1$ whenever $a+b>1$.

Generalized effect algebras and equivalent difference posets were considered as generalizations of effect algebras (D-posets) without the unit, see [3, 4].

A "large associativity" (also for infinite number of elements) of the partial operation $\oplus$ was studied by Riečanová [10] in the context of abelian RI-posets for complete lattices, by Ji [8] for orthocomplete effect algebras and by Tkadlec [11] in the context of effect algebras.

The distributivity-like properties of suprema and infima (possibly infinite) with respect to partial operations $\oplus$ and $\ominus$ and vice versa were studied by

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Bennett and Foulis [1] in the context of effect algebras (sometime assuming that they form a lattice), by Chovanec and Kôpka [2] in the context of D-posets (for two-element sets assuming that the D-posets form a lattice) and by Tkadlec [11] in the context of effect algebras.

We generalize these results for generalized effect algebras and present examples showing that these results cannot be improved to obtain distributivity (associativity, resp.) in all cases.

Our results can be useful in the study of generalized effect algebras (quantum and fuzzy structures)—see, e.g., [4].

2 Basic notions and properties

Let us start with a review of basic notions and properties.

2.1 Definition. A *generalized effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0})$ such that E is a set, $\oplus$ is a partial binary operation on E and $\mathbf{0} \in E$ such that for every $a, b, c \in E$ the following conditions hold:
(1) $a \oplus b = b \oplus a$, if one side exists (commutativity);
(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if one side exists (associativity);
(3) $b = c$ whenever $a \oplus b = a \oplus c$ (cancellation law);
(4) $a = b = \mathbf{0}$ whenever $a \oplus b = \mathbf{0}$ (positivity);
(5) $a \oplus \mathbf{0} = a$.

For simplicity, we will use the notation E for a generalized effect algebra. An *orthogonality* relation $\perp$ on a generalized effect algebra E is defined by $a \perp b$ if $a \oplus b$ exists. A partial ordering on a generalized effect algebra E is defined by $a \leq b$ if there is a $c \in E$ such that $b = a \oplus c$. Such an element c is unique (if it exists) and is denoted by $b \ominus a$. With respect to this partial ordering, $\mathbf{0}$ is the least element of E. A generalized effect algebra is an effect algebra if and only if it contains the greatest element (denoted by $\mathbf{1}$). The orthosupplement of an element a in an effect algebra is $a' = \mathbf{1} \ominus a$. See, e.g., [4].

Let us present an example of a generalized effect algebra that is not an effect algebra.

2.2 Example. Let $[0, 1)$ be the interval of real numbers, $\oplus$ be the partial binary operation on $[0, 1)$ defined by $a \oplus b = a + b$ if $a + b < 1$. Then $([0, 1), \oplus, 0)$ is a generalized effect algebra that si not an effect algebra. Its partial ordering is the standard ordering of real numbers on $[0, 1)$ and $a \ominus b = a - b$ (for $b \leq a$).

2.3 Definition. Let E be a generalized effect algebra. A system $(a_i)_{i \in I}$ of elements of E is *orthogonal* if $\bigoplus_{i \in F} a_i$ is defined for every finite set $F \subseteq I$. A *majorant* of an orthogonal system is an upper bound of all its finite sums. The *sum* $\bigoplus_{i \in I} a_i$ of an orthogonal system $(a_i)_{i \in I}$ is its least majorant (if it exists).

A finite system is orthogonal if and only if the sum of all its elements is defined. Every subsystem of an orthogonal system is orthogonal. The empty system is orthogonal and its sum is the least element $\mathbf{0}$. Every pair of elements in an orthogonal system is orthogonal. On the other hand there are nonorthogonal systems of pairwise orthogonal elements if (and only if) the generalized effect algebra does not form an orthomodular poset.

Let us summarize some properties of the operations $\oplus$ and $\ominus$ showing that these partial operations behave very much like the real operations $+$ and $-$. The basic difference is that we have to take care whether they are defined.

2.4 Lemma. *Let E be a generalized effect algebra, $a, b, c, a_i \in E$, $i \in I$, I is finite:*

(1) *If $b = \bigoplus_{i \in I} a_i$ and $J \subseteq I$ then $b \geq \bigoplus_{i \in J} a_i$ and $b \ominus \bigoplus_{i \in J} a_i = \bigoplus_{i \in I \setminus J} a_i$. In particular, $(a \oplus b) \ominus b = a$ whenever $a \perp b$.*

(2) *If $a \leq b$ then $a \oplus (b \ominus a) = b$, $b \ominus (b \ominus a) = a$.*

(3) *$a \oplus b \leq c$ if and only if $a \leq c \ominus b$.*

(4) *If $a \leq b \perp c$ then $b \oplus c = a \oplus c \oplus (b \ominus a)$, i.e., $a \oplus c \leq b \oplus c$ and $(b \oplus c) \ominus a = c \oplus (b \ominus a)$.*

(5) *If $a \leq b \leq c$ then $c \ominus a = (b \ominus a) \oplus (c \ominus b)$, i.e., $b \ominus a \leq c \ominus a$ and $c \ominus b \leq c \ominus a$.*

(6) *$c \ominus (a \oplus b) = (c \ominus a) \ominus b$ whenever one side exists.*

Proof. (1) It is a consequence of the commutativity and associativity of $\oplus$ and of the definition of $\leq$ and $\ominus$.

(2) The first equality is the definition of $b \ominus a$, the second follows using part (1).

(3) $a \oplus b \leq c$ if and only if there is a $d \in E$ such that $c = a \oplus b \oplus d$. According to part (1), this is equivalent to $c \ominus b = a \oplus d$ for some $d \in E$, and this is equivalent to $a \leq c \ominus b$.

(4) $b \oplus c = (a \oplus (b \ominus a)) \oplus c = a \oplus c \oplus (b \ominus a)$.

(5) Since $a \oplus (c \ominus a) = c = b \oplus (c \ominus b) = a \oplus (b \ominus a) \oplus (c \ominus b)$, using the cancellation law we obtain $c \ominus a = (b \ominus a) \oplus (c \ominus b)$.

(6) $a \oplus b \leq c$ if and only if there is a $d \in E$ such that $c = a \oplus b \oplus d$; this is equivalent to $b \leq c \ominus a$. Then $c = a \oplus (c \ominus a) = a \oplus b \oplus (c \ominus a) \ominus b$. According to part (1), $c \ominus (a \oplus b) = (c \ominus a) \ominus b$. □

To simplify some notations we will use sets of elements instead of elements as arguments of relations and operations in a usual way. E.g., if a is an element and B is a set of elements of a generalized effect algebra then by $a \leq B$ we mean that $a \leq b$ for every $b \in B$ and by $a \oplus B$ we mean the set $\{a \oplus b \colon b \in B\}$.

Suprema preserve orthogonality in effect algebras. This is not true in general in generalized effect algebras.

2.5 Lemma. *Let E be a generalized effect algebra, $a \in E$, $B \subseteq E$, $a \perp B$ and $\bigvee B$ exists. Then $a \perp \bigvee B$ if and only if $a \oplus B$ has an upper bound.*

Proof. If $a \perp \bigvee B$ then $a \oplus \bigvee B$ is an upper bound of $a \oplus B$.

Let c be an upper bound of $a \oplus B$. Then $c \geq a \oplus B$ and therefore $c \ominus a \geq B$, hence $c \ominus a \geq \bigvee B$ and therefore $c \geq a \oplus \bigvee B$, i.e., $a \perp \bigvee B$. □

2.6 Example. Let $E = ([0, 1), \oplus, 0)$ be the generalized effect algebra from Example 2.2, $a = \frac{1}{2}$, $B = [0, \frac{1}{2})$. Then $a \perp B$, $\bigvee B = \frac{1}{2}$ exists but $a \not\perp \bigvee B$ ($a \oplus B = [\frac{1}{2}, 1)$ does not have an upper bound).

3 Associativity

The partial operation $\oplus$ is associative considering finite sums. We will consider "large associativity" including also infinite sums, i.e. (using the commutativity of $\oplus$), $\bigoplus_{i\in I} a_i = \bigoplus_{j\in J} \bigoplus_{i\in I_j} a_i$ for an orthogonal system $(a_i)_{i\in I}$ where I is a disjoint union of I_j, $j \in J$. This was proved for two-element set J (it might be easily generalized for finite J) by Riečanová [10, Theorem 1.6 (iv)] in the context of abelian RI-posets for complete lattices with the assumption that the right side exists, by Ji [8, Lemma 3.2] for orthocomplete effect algebras (the existence of all considered sums is ensured) and by Tkadlec [11, Theorem 4.2] for effect algebras (assuming the right side exists).

Let us start with properties of "disjoint" subsums of an orthogonal system.

3.1 Proposition. *Let E be a generalized effect algebra, $(a_i)_{i\in I}$ be an orthogonal system in E, I be a disjoint union of I_j, $j \in J$, $b_j = \bigoplus_{i\in I_j} a_i$ exists for every $j \in J$.*

(1) *If the system $(a_i)_{i\in I}$ has a majorant then the system $(b_j)_{j\in J}$ is orthogonal.*

(2) *If the system $(b_j)_{j\in J}$ is orthogonal then the set of its majorants is the set of majorants of the system $(a_i)_{i\in I}$.*

Proof. Let us denote by $\mathcal{F}$ the family of finite subsets of I and, for every $j \in J$ and every $F \in \mathcal{F}$, $b_{j,F} = \bigoplus_{i\in I_j\cap F} a_i$ and $J_F = \{j \in J : I_j \cap F \neq \emptyset\}$.

(1) Let c be a majorant of $(a_i)_{i\in I}$, $G \subseteq J$ be finite, $F_j \subseteq I_j$ be finite for every $j \in G$. Then $c \geq \bigoplus_{i\in\bigcup\{F_j :\, j\in G\}} a_i = \bigoplus_{j\in G} b_{j,F_j}$. For every $k \in G$, we consecutively obtain $c \ominus \bigoplus_{j\in G\setminus\{k\}} b_{j,F_j} \geq b_{k,F_k}$, $c \ominus \bigoplus_{j\in G\setminus\{k\}} b_{j,F_j} \geq b_k$, $c \geq b_k \oplus \bigoplus_{j\in G\setminus\{k\}} b_{j,F_j}$. Repeating this procedure, we obtain $c \geq \bigoplus_{j\in G} b_j$, i.e., the system $(b_j)_{j\in J}$ is orthogonal and c is its majorant.

(2) Let c be a majorant of $(b_j)_{j\in J}$. Then, for every $F \in \mathcal{F}$, $\bigoplus_{i\in F} a_i = \bigoplus_{j\in J_F} b_{j,F} \leq \bigoplus_{j\in J_F} b_j \leq c$, i.e., c is a majorant of $(a_i)_{i\in I}$. The reverse implication was proved in the proof of part (1). $\square$

The following example shows that sums of subsystems of an orthogonal system need not be orthogonal (i.e., the assumption at the part (1) of Proposition 3.1 cannot be omitted).

3.2 Example. Let $E = ([0,1), \oplus, 0)$ be the generalized effect algebra from Example 2.2, I be the set of natural numbers, $a_i = 2^{-i}$ for every $i \in I$, $I_1 = \{1\}$, $I_2 = I \setminus I_1$. Then $(a_i)_{i\in I}$ is an orthogonal system, $b_1 = \bigoplus_{i\in I_1} a_i = a_1 = \frac{1}{2}$, $b_2 = \bigoplus_{i\in I_2} a_i = \frac{1}{2}$, $(b_1, b_2) = (\frac{1}{2}, \frac{1}{2})$ is not an orthogonal system.

The following example shows that the assumption at the part (1) of Proposition 3.1 is not necessary.

3.3 Example. Let $E = ([0,1), \oplus, 0)$ be the generalized effect algebra from Example 2.2, $I = J$ be the set of natural numbers, $a_i = 2^{-i}$ for every $i \in I$, $I_j = \{j\}$ for every $j \in J$. Then the system $(a_i)_{i\in I}$ does not have a majorant ($\sum_{i\in I} a_i = 1 \notin [0,1)$) but it is orthogonal.

3.4 Theorem. *Let E be a generalized effect algebra, $(a_i)_{i\in I}$ be an orthogonal system in E, I be a disjoint union of I_j, $j \in J$, K be a subset of J.*

(1) *If $\bigoplus_{j\in J} \bigoplus_{i\in I_j} a_i$ exists then it is equal to $\bigoplus_{i\in I} a_i$.*

(2) *If* $\bigoplus_{i\in I} a_i$ *and* $\bigoplus_{j\in K}\bigoplus_{i\in I_j} a_i$ *exist then* $\bigoplus_{i\in I} a_i \ominus \bigoplus_{j\in K}\bigoplus_{i\in I_j} a_i$ *is a minimal majorant of* $(a_i)_{i\in\bigcup\{I_j\,:\,j\in J\setminus K\}}$*, i.e., a minimal majorant of the system* $\big(\bigoplus_{i\in I_j} a_i\big)_{j\in J\setminus K}$ *if its sums exist.*

Proof. (1) It is a consequence of Proposition 3.1.

(2) According to part (1), $\bigoplus_{j\in K}\bigoplus_{i\in I_j} a_i = \bigoplus_{i\in\bigcup\{I_j\,:\,j\in K\}} a_i$, hence, without a loss of generality we may (and will) assume that $J=\{1,2\}$ and $K=\{1\}$. Let us denote by $\mathcal{F}$ the family of finite subsets of I, $a_F=\bigoplus_{i\in F} a_i$, $a=\bigoplus_{i\in I} a_i$, $b_{j,F}=\bigoplus_{i\in I_j\cap F} a_i$, for every $F\in\mathcal{F}$ and every $j\in J$, $b_1=\bigoplus_{i\in I_1} a_i$.

Let $F_1\subseteq I_1$ and $F_2\subseteq I_2$ be finite. We consecutively obtain $b_{1,F_1}\oplus b_{2,F_2} = a_{F_1\cup F_2}\le a$, $b_{1,F_1}\le a\ominus b_{2,F_2}$, $b_1\le a\ominus b_{2,F_2}$, $b_{2,F_2}\le a\ominus b_1$. Hence $a\ominus b_1$ is a majorant of $(a_i)_{i\in I_2}$. Let c be a majorant of $(a_i)_{i\in I_2}$ with $c\le a\ominus b_1$. Then $a\ominus\big((a\ominus b_1)\ominus c\big) = a\ominus\big(a\ominus(b_1\oplus c)\big) = b_1\oplus c\ge b_{1,F}\oplus b_{2,F}=a_F$ for every $F\in\mathcal{F}$ and therefore $a\ominus\big((a\ominus b_1)\ominus c\big)\ge a$, i.e., $c=a\ominus b_1$. Hence, $a\ominus b_1$ is a minimal majorant of $(a_i)_{i\in I_2}$. □

Let us discuss part (2) of Theorem 3.4. We have shown in Example 3.2 that the system $(\bigoplus_{i\in I_j} a_i)_{j\in J}$ need not be orthogonal (it is orthogonal in effect algebras). Moreover, some summable orthogonal systems might be divided to unsummable subsystems [11, Example 4.3] (even in Boolean algebras) and the minimal majorant need not be the sum [11, Example 4.4] (even in orthomodular posets).

4 Distributivity-like properties

The following Theorems 4.1, 4.2 and 4.3 describe distributivity-like properties of $\oplus$ and $\ominus$ with respect to (possibly infinite) suprema and infima and vice versa. All these theorems follow the same pattern and are generalizations of [11, Theorems 3.1, 3.3 and 3.2] stated for effect algebras that generalize and cover some previous results: Theorem 4.1 is a generalization of [2, Propositions 2.3 and 2.4] proved for lattices and 2-element sets (in the context of D-posets). Theorem 4.2 (2) is a generalization of [1, Theorem 3.2] stated for lattices. Theorem 4.2 (3) is a generalization of [1, Theorem 2.2] proved for effect algebras. Theorem 4.3 is a generalization of [2, Propositions 2.6 and 2.9] proved for lattices and 2-element sets (in the context of D-posets). Theorem 4.3 (1) is a generalization of [1, Corollary 2.3] proved for effect algebras.

Let us start with generalizations of de Morgan laws, that might be formulated in an effect algebra (or a bounded poset) E as follows: For every $B\subseteq E$ we have $(\bigvee B)'=\bigwedge B'$ ($(\bigwedge B)'=\bigvee B'$, resp.) if one side exists. Equivalent equalities using the partial operation $\ominus$ are: $\mathbf{1}\ominus\bigvee B=\bigwedge(\mathbf{1}\ominus B)$, $\mathbf{1}\ominus\bigwedge B=\bigvee(\mathbf{1}\ominus B)$. Since a generalized effect algebra need not have the greatest element $\mathbf{1}$, we replace it by an upper bound.

4.1 Theorem. *Let E be a generalized effect algebra, $a\in E$, $B\subseteq E$, $B\le a$.*

(1) *If* $\bigvee B$ *exists then* $a\ominus\bigvee B=\bigwedge(a\ominus B)$.

(2) *If* $\bigwedge(a\ominus B)$ *exists then* $a\ominus\bigwedge(a\ominus B)$ *is a minimal upper bound of* B.

(3) *If* $\bigvee(a\ominus B)$ *exists then* $a\ominus\bigwedge B=\bigvee(a\ominus B)$.

(4) *If* $\bigwedge B$ *exists then* $a\ominus\bigwedge B$ *is a minimal upper bound of* $a\ominus B$.

Proof. (1) We have $B \le \bigvee B \le a$ and therefore $a \ominus \bigvee B \le a \ominus B$, i.e., $a \ominus \bigvee B$ is a lower bound of $a \ominus B$. Let $c \in E$ be a lower bound of $a \ominus B$. We have $c \le a \ominus B$ and therefore $B \le a \ominus c$, hence $\bigvee B \le a \ominus c$ and therefore $c \le a \ominus \bigvee B$, i.e., $a \ominus \bigvee B$ is the greatest lower bound of $a \ominus B$.

(2) We have $\bigwedge(a \ominus B) \le a \ominus B$ and therefore $B \le a \ominus \bigwedge(a \ominus B)$, i.e., $a \ominus \bigwedge(a \ominus B)$ is an upper bound of B. Let $c \in E$ be an upper bound of B such that $c \le a \ominus \bigwedge(a \ominus B)$. We have $B \le c \le a$ and therefore $a \ominus c \le a \ominus B$, hence $a \ominus c \le \bigwedge(a \ominus B)$ and therefore $c \ge a \ominus \bigwedge(a \ominus B)$, i.e., $a \ominus \bigwedge(a \ominus B)$ is a minimal upper bound of B.

(3), (4) follows from parts (1) and (2) if we replace B by $a \ominus B$. $\square$

4.2 Theorem. *Let E be a generalized effect algebra, $a \in E$, $B \subseteq E$, $a \perp B$.*

(1) *If $\bigwedge(a \oplus B)$ exists then $a \oplus \bigwedge B = \bigwedge(a \oplus B)$.*

(2) *If $\bigwedge B$ exists then $a \oplus \bigwedge B$ is a maximal lower bound of $a \oplus B$.*

(3) *If $\bigvee B$ exists and $a \perp \bigvee B$ (or, equivalently, $a \oplus B$ has an upper bound) then $a \oplus \bigvee B = \bigvee(a \oplus B)$.*

(4) *If $\bigvee(a \oplus B)$ exists then $\bigvee(a \oplus B) \ominus a$ is a minimal upper bound of B.*

Proof. (1) We have $a \le \bigwedge(a \oplus B) \le a \oplus B$ and therefore $\bigwedge(a \oplus B) \ominus a \le B$, i.e., $\bigwedge(a \oplus B) \ominus a$ is a lower bound of B. Let $c \in E$ be a lower bound of B. We have $c \le B$ and therefore $a \oplus c \le a \oplus B$, hence $a \oplus c \le \bigwedge(a \oplus B)$ and therefore $c \le \bigwedge(a \oplus B) \ominus a$. Hence $\bigwedge(a \oplus B) \ominus a = \bigwedge B$ and therefore $\bigwedge(a \oplus B) = a \oplus \bigwedge B$.

(2) We have $\bigwedge B \le B$ and therefore $a \oplus \bigwedge B \le a \oplus B$, i.e., $a \oplus \bigwedge B$ is a lower bound of $a \oplus B$. Let $c \in E$ be a lower bound of $a \oplus B$ such that $a \oplus \bigwedge B \le c$. We have $a \le c \le a \oplus B$ and therefore $c \ominus a \le B$, hence $c \ominus a \le \bigwedge B$ and therefore $c \le a \oplus \bigwedge B$. Hence, $a \oplus \bigwedge B$ is a maximal lower bound of $a \oplus B$.

(3) $a \oplus \bigvee B$ is an upper bound of $a \oplus B$. Let $c \in E$ be an upper bound of $a \oplus B$. We have $a \oplus B \le c$ and therefore $B \le c \ominus a$, hence $\bigvee B \le c \ominus a$ and therefore $a \oplus \bigvee B \le c$. Hence, $a \oplus \bigvee B$ is the least upper bound of $a \oplus B$.

(4) We have $a \oplus B \le \bigvee(a \oplus B)$ and therefore $B \le \bigvee(a \oplus B) \ominus a$, i.e., $\bigvee(a \oplus B) \ominus a$ is an upper bound of B. Let $c \in E$ be an upper bound of B such that $c \le \bigvee(a \oplus B) \ominus a$. We have $a \oplus B \le a \oplus c$ and therefore $\bigvee(a \oplus B) \le a \oplus c$, hence $\bigvee(a \oplus B) \ominus a \le c$. Hence, $\bigvee(a \oplus B) \ominus a$ is a minimal upper bound of B. $\square$

The following theorem is a reformulation of Theorem 4.2 for $a \oplus B$ instead of B.

4.3 Theorem. *Let E be a generalized effect algebra, $a \in E$, $B \subseteq E$, $a \le B$.*

(1) *If $\bigwedge B$ exists then $\bigwedge B \ominus a = \bigwedge(B \ominus a)$.*

(2) *If $\bigwedge(B \ominus a)$ exists then $a \oplus \bigwedge(B \ominus a)$ is a maximal lower bound of B.*

(3) *If $\bigvee(B \ominus a)$ exists and $a \perp \bigvee(B \ominus a)$ (or, equivalently, B has an upper bound) then $\bigvee B \ominus a = \bigvee(B \ominus a)$.*

(4) *If $\bigvee B$ exists then $\bigvee B \ominus a$ is a minimal upper bound of $B \ominus a$.*

Minimal upper bounds (maximal lower bounds, resp.) in Theorems 4.1, 4.2 and 4.3 could not be replaced by suprema (infima, resp.) in general even for orthomodular posets. The following example shows that for Theorem 4.1 (2), other examples might be derived easily.

4.4 Example. Let $X = \{1, 2, 3, 4, 5, 6\}$, E be the family of even-element subsets of X, $\oplus$ be the union of disjoint sets. $(E, \oplus, \emptyset)$ is a generalized effect algebra (forms an orthomodular poset with the greatest element X), the partial ordering is the inclusion. Then $a = \{1,2,3,4\}$, $B = \big\{\{1,2\},\{2,3\}\big\}$ fulfills the assumptions of Theorem 4.1 (2), $\bigvee B$ does not exist.

Example 2.6 shows that the the condition $a \perp \bigvee B$ (or, equivalently, the existence of an upper bound of $a \oplus B$) cannot be omitted in Theorem 4.2 (3). (An analogous example might be derived for Theorem 4.3 (3).)

Theorem 4.1 and part of Theorems 4.2 and 4.3 might be significantly simplified in generalized effect algebras where a minimal upper bound has to be a supremum, e.g., in lattice generalized effect algebras.

4.5 Corollary. *Let E be a lattice generalized effect algebra, $a \in E$, $B \subseteq E$.*

(1) *If $B \le a$ then $a \ominus \bigvee B = \bigwedge(a \ominus B)$ and $a \ominus \bigwedge B = \bigvee(a \ominus B)$ whenever one side of the respective equality exists.*

(2) *If $a \perp B$ then $a \oplus \bigwedge B = \bigwedge(a \oplus B)$ whenever one side of the equality exists.*

(3) *If $a \le B$ then $\bigwedge B \ominus a = \bigwedge(B \ominus a)$ whenever one side of the equality exists.*

4.6 Corollary. *Let E be a generalized effect algebra, $a, b \in E$ such that $a \perp b$ and $a \vee b$ exists. Then $a \wedge b$ exists and $a \oplus b = (a \vee b) \oplus (a \wedge b)$. In particular, $a \oplus b \ge a \vee b$ and the equality is valid if and only if $a \wedge b = \mathbf{0}$.*

Proof. It follows from Theorem 4.1, part (1), for $\{a, b\} \le a \oplus b$. □

Let us remark that the inequality $a \oplus b \ge a \vee b$ in the above statement is obvious.

If we put $a = \bigvee B$ in Theorem 4.1 (1), we obtain $\bigwedge(a \ominus B) = \mathbf{0}$. If we put $a = \bigwedge B$ in Theorem 4.3 (1), we obtain $\bigwedge(B \ominus a) = \mathbf{0}$. Let us present stronger results.

4.7 Theorem. *Let E be a generalized effect algebra, $a \in E$, $B \subseteq E$.*

(1) *$\bigwedge(a \ominus B) = \mathbf{0}$ if and only if a is a minimal upper bound of B.*

(2) *$\bigwedge(B \ominus a) = \mathbf{0}$ if and only if a is a maximal lower bound of B.*

Proof. (1) For every $c \in E$, $c \le a \ominus B$ if and only if $B \le a \ominus c$. Hence, $\bigwedge(a \ominus B) = \mathbf{0}$ if and only if there is no smaller upper bound of B than a.

(2) For every $c \in E$, $c \le B \ominus a$ if and only if $a \oplus c \le B$. Hence, $\bigwedge(B \ominus a) = \mathbf{0}$ if and only if there is no greater lower bound of B than a. □

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Statements and Declarations

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