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Josef Tkadlec

A NOTE ON DISTRIBUTIVITY IN ORTHOPOSETS

It is well-known that distributive ortholattices are Boolean (i.e., they fulfil the condition $a \wedge b = 0 \Rightarrow b \leq a'$). In this note we formulate some distributivity-like condition valid in Boolean orthoposets and prove that Boolean ω-orthocomplete poset has to be orthomodular. Our results generalize results of Klukowski [2, 3] and, also, might find application in the axiomatics of quantum theories (see [1, 5]).

Notions and results

Let us first review basic notions as we shall use them throughout the paper.

1. Definition. An *orthoposet* is a triple $(P, \leq, ')$ such that

(1) $(P, \leq)$ is a partially ordered set with a least element, 0, and a greatest element, 1,

(2) $' : P \to P$ is an *orthocomplementation*, i.e., (i) $a'' = a$, (ii) $a \leq b \Rightarrow b' \leq a'$, (iii) $a \wedge a' = 0$ for every $a, b \in P$.

An orthoposet $(P, \leq, ')$ is called an *ω-orthocomplete poset* if $a \vee b$ exists in P for every $a, b \in P$ such that $a \leq b'$, and it is called an *orthocomplete poset* if $\bigvee S$ exists in P for every $S \subset P$ such that $s_1 \leq s_2'$ for any pair $s_1, s_2 \in S$.

Further, an ω-orthocomplete poset is called *orthomodular* if $b = a \vee (b \wedge a')$ for every $a, b \in P$ such that $a \leq b$.

An orthoposet $(P, \leq, ')$ is called *Boolean* ([2]) if the condition $a \wedge b = 0$ implies $b \leq a'$.

For any orthoposet $(P, \leq, ')$, let us write $[a, b] = \{c \in P;\ a \leq c \leq b\}$ for every $a, b \in P$, $S_1 \leq S_2$ $(S_1, S_2 \subset P)$ if $s_1 \leq s_2$ for every $s_1 \in S_1$ and for every $s_2 \in S_2$, $s \leq S$ $(s \in P,\ S \subset P)$ if $\{s\} \leq S$, $S_1 \wedge \ldots \wedge S_n = \{s_1 \wedge \ldots \wedge s_n;\ s_1 \in S_1, \ldots, s_n \in S_n\}$ $(S_1, \ldots, S_n \subset P)$.

2. Lemma. *Suppose that $(P, \leq, ')$ is a Boolean orthoposet and that $S_1 \cup \ldots \cup S_n \subset P$. Let us write*

$$L = \bigcup \{[0, s_1] \cap \ldots \cap [0, s_n];\ (s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n\},$$

$$U = \bigcup_{k=1}^{n} \bigcap_{s_k \in S_k} [s_k, 1].$$

Then $L \leq U$ and $l \leq u$ for every $l \leq U$ and for every $u \geq L$.

Proof. The inequality $L \leq U$ is evident. Suppose that $l \not\leq u$. Then there is an $a \in P \setminus \{0\}$ such that $a \leq l, u'$. Since $l \leq \bigcap_{s_1 \in S_1} [s_1, 1]$, we obtain $a' \notin \bigcap_{s_1 \in S_1} [s_1, 1]$. It means that $s_1 \not\leq a'$ for some $s_1 \in S_1$. Hence, there is an $a_1 \in P \setminus \{0\}$ such that $a_1 \leq s_1, a, u'$. Proceeding by induction, we obtain an $a_n \in P \setminus \{0\}$ such that $a_n \leq s_1, \ldots, s_n, u'$ for some $s_1 \in S_1, \ldots, s_n \in S_n$. Therefore we have $a_n \in L \leq u$ and $a_n \leq u \wedge u' = 0$, which is a contradiction.

3. Theorem. *Suppose that $(P, \leq, ')$ is a Boolean orthoposet and that $S_1 \cup \ldots \cup S_n \subset P$ such that $S_1 \wedge \ldots \wedge S_n$, $\bigvee S_1, \ldots, \bigvee S_n$ exist in $(P, \leq, ')$. Then*

$$\bigvee (S_1 \wedge \ldots \wedge S_n) = \left(\bigvee S_1 \right) \wedge \ldots \wedge \left(\bigvee S_n \right)$$

if at least one side of this equality exists.

Proof. The left side of this equality exists if and only if $\bigvee L$ exists (L taken from Lemma 2) and both expressions are equal. The right side of this equality exists if and only if $\bigwedge U$ exists (U taken from Lemma 2) and both expressions are equal. According to Lemma 2, $\bigvee L$ exists if and only if $\bigwedge U$ exists and then $\bigvee L = \bigwedge U$.

4. Corollary. *Every Boolean ω-orthocomplete poset is orthomodular.*

5. Corollary. *Every Boolean ortholattice is a Boolean algebra.*

Let us recall that an orthoposet $(P, \leq, ')$ is called *atomic* if for any $b \in P \setminus \{0\}$ there is an $a \in P \setminus \{0\}$ such that $[0, a] = \{0, a\}$ (i.e., a is an *atom*) and $a \leq b$.

6. Theorem. *Every atomic Boolean orthocomplete poset is a Boolean algebra.*

Proof. It follows from Corollary 4 and from [2], Theorem 2.

Let us note that Boolean orthoposets are *concrete* (i.e., they are set-representable in such a manner that the supremum of a finite number of mutually disjoint sets (if it exists) is the set-theoretic union, see [4] for Boolean orthomodular posets — the orthomodularity was not used in the proof). As the following simple example shows, not every Boolean orthoposet is orthomodular.

7. Example. Let X be a four-element set and let $(P, \leq, ')$ be a triple such that P consists of $\emptyset$, one-element subsets of X and set-complements of these sets, $\leq$ means the inclusion in X and $'$ the set-theoretic complementation. Then $(P, \leq, ')$ is a Boolean orthoposet which is not orthomodular.

Finally, let us state for comparison results analogous to Theorem 3. We shall need the following definition.

8. Definition. Let $(P, \leq, ')$ be an orthoposet. Then elements $a, b \in P$ are called *compatible* (denoted by $a \leftrightarrow b$) if there are $a_1, b_1, c \in P$ such that $a = a_1 \vee c$, $b = b_1 \vee c$ and $a_1 \leq b_1'$, $a_1 \leq c'$, $b_1 \leq c'$.

9. Proposition. *Suppose that $(P, \leq, ')$ is an orthomodular lattice and that $S_1 \cup \ldots \cup S_n \subset P$ such that $S_1 \wedge \ldots \wedge S_n$, $\bigvee S_1, \ldots, \bigvee S_n$ exists in $(P, \leq, ')$ and such that $s_i \leftrightarrow s_j$ for every pair $s_i \in S_i, s_j \in S_j$, $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then*

$$\bigvee (S_1 \wedge \ldots \wedge S_n) = \left(\bigvee S_1\right) \wedge \ldots \wedge \left(\bigvee S_n\right)$$

if the right side of this equality exists.

Proof. It follows from [5], Proposition 1.3.10, if we proceed by the induction.

10. Proposition. *Suppose that $(P, \leq, ')$ is an orthomodular poset and that $\{s_1\} \cup S_2 \subset P$ such that $\{s_1\} \wedge S_2$, $\bigvee \{s_1\}, \bigvee S_2$ exist in $(P, \leq, ')$ and such that $s_1 \leftrightarrow s_2$ for every $s_2 \in S_2$. Then*

$$\bigvee (\{s_1\} \wedge S_2) = \left(\bigvee \{s_1\}\right) \wedge \left(\bigvee S_2\right)$$

if the left side of this equality exists.

Proof. See [1], Lemma 3.7.

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Department of Mathematics, Technical University of Prague, 166 27 Praha, Czechoslovakia

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