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BOOLEAN ORTHOPOSETS AND TWO-VALUED STATES ON THEM

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A Boolean orthoposet (see e.g. [2]) is the orthoposet P fulfilling the following condition: If $a, b \in P$ and $a \wedge b = 0$ then $a \perp b$. This condition seems to be a sound generalization of distributivity in orthoposets (see e.g. [8]). Also, the class of (orthomodular) Boolean orthoposets may play an interesting role in quantum logic theory. This class is wide enough (see [4, 3]) and, on the other hand, enjoys some properties of Boolean algebras [4, 8, 5]. In quantum logic theory an important role is played by so-called Jauch–Piron states [1, 6, 7]. In this paper we clarify the connection between Boolean orthoposets and orthoposets with "enough" two-valued Jauch–Piron states. Further, we obtain a characterization of Boolean orthoposets in terms of two-valued states.

1. Preliminaries and basic notions

1.1. Definition. An *orthoposet* is a triple $(P, \leq, ')$ such that

(1) $(P, \leq)$ is a partially ordered set with a least and a greatest elements 0, 1,

(2) $' : P \to P$ is an *orthocomplementation*, i.e., (i) $a'' = a$, (ii) $a \leq b \Rightarrow b' \leq a'$, (iii) $a \vee a' = 1$ for every $a, b \in P$.

In the sequel we will shortly write P instead of $(P, \leq, ')$ and reserve the letter P for orthoposets.

1.2. Definition. Elements a, b of P are called *orthogonal* (denoted by $a \perp b$) if $a \leq b'$.

An orthoposet P is called *Boolean* if $a \perp b$ whenever $a \wedge b = 0$, and it is called *orthomodular* if $a \vee b$ exists whenever $a \perp b$ and $b = a \vee (b \wedge a')$ whenever $a \leq b$.

For the further use, let us note that in Boolean orthoposets the condition $a \not\perp b$ (i.e., $a \not\leq b'$) implies that there is a $c \neq 0$ such that $c \leq a, b'$.

1.3. Definition. A *concrete orthoposet* is a triple $(\mathcal{P}, \subset, ^c)$ where $\mathcal{P} \subset \exp X$ for some $X \neq \emptyset$ such that

(1) $\emptyset \in \mathcal{P}$,

(2) $A^c = X \setminus A \in \mathcal{P}$ whenever $A \in \mathcal{P}$,

(3) $\bigcup \mathcal{F} \in \mathcal{P}$ for every finite family $\mathcal{F} \subset \mathcal{P}$ of mutually disjoint elements such that $\bigvee \mathcal{F}$ exists in $(\mathcal{P}, \subset)$.

Let us observe that a concrete orthoposet is indeed an orthoposet and that a concrete orthoposet is orthomodular if and only if $A \cup B \in \mathcal{P}$ for every $A, B \in \mathcal{P}$ with $A \cap B = \emptyset$. For every set $X \neq \emptyset$ and for every family $\mathcal{F}$ of subsets of X there is the least orthomodular poset $\mathcal{P} \supset \mathcal{F}$. We say that $\mathcal{P}$ is the orthomodular poset *generated* by $\mathcal{F}$. Every element of $\mathcal{P}$ can be obtained from elements of $\mathcal{F}$ using finitely many operations of set-theoretic complement in X and of union of two disjoint elements.

A central role in this paper will be played by states (measures) on orthoposets.

1.4. Definition. A *state* on P is a mapping $s : P \to [0,1]$ such that
(1) $s(1) = 1$,
(2) $s(a) \leq s(b)$ whenever $a \leq b$,
(3) $s(\bigvee F) = \sum_{a \in F} s(a)$ for every finite set $F \in P$ of mutually orthogonal elements such that $\bigvee F$ exists in $(P, \leq)$.

Let us note that $1 = s(1) = s(a \vee a') = s(a) + s(a')$ for every $a \in P$. In particular, $s(0) = 0$. Hence, a two-valued state is a state with values in the set $\{0,1\}$.

1.5. Definition. A state s on P is called *Jauch–Piron* if for every pair $a, b \in P$ with $s(a) = s(b) = 1$ there is a $c \in P$ with $s(c) = 1$ such that $c \leq a, b$.

1.6. Definition. A set S of (not necessarilly all) states on an orthoposet P is called:
unital if for every $a \in P \setminus \{0\}$ there is a state $s \in S$ such that $s(a) = 1$;
full if for every pair $a, b \in P$ with $a \not\leq b$ there is a state $s \in S$ such that $s(a) \not\leq s(b)$.

Let us note that every full set of two-valued states is unital and that an orthoposet has a full set of two-valued states if and only if it has a concrete representation (Stone-like representation) — see [9].

2. Characterization of Boolean orthoposets

In this section we will characterize Boolean orthoposets by means of two-valued states.

2.1. Theorem. *The set of two-valued states on a Boolean orthoposet is full.*

Proof: See [4], inclusion $\mathcal{C}_3 \subset \mathcal{C}$ in Theorem 3.1. (It is proved for orthomodular Boolean orthoposets there but the orthomodularity was not used in the proof.)

2.2. Theorem. *Let P be an orthoposet. The following two properties are equivalent:*
(1) *P is a Boolean orthoposet.*
(2) *The orthoposet P has a unital set of two-valued states and every unital set of two-valued states on P is full.*

Proof: (1)$\Rightarrow$(2). According to Theorem 2.1, the set of two-valued states on P is full, hence it is unital. Let us suppose that S is a unital set of two-valued states on P and that $a \not\leq b$. There is a $c \in P \setminus \{0\}$ such that $c \leq a, b'$ and a two-valued state $s \in S$ such that $s(c) = 1$. Thus, $s(a) = s(b') = 1$ and $s(a) = 1 \not\leq 0 = 1 - s(b') = s(b)$.

(2)⇒(1). Let $a, b \in P$ be such that $a \wedge b = 0$, i.e., for every $c \in P \setminus \{0\}$ we have either $c \not\leq a$ or $c \not\leq b$. According to our assumptions, the set of two-valued states on P is full. Hence for every $c \in P \setminus \{0\}$ there is a two-valued state s on P such that either $s(c) \not\leq s(a)$ or $s(c) \not\leq s(b)$. Thus, the set S of two-valued states s on P such that either $s(a) = 0$ or $s(b) = 0$ is unital and, according to our assumptions, full. Since $s(a) \leq s(b') = 1 - s(b)$ for every $s \in S$, we obtain $a \leq b'$, hence $a \perp b$. The proof is complete.

3. Boolean orthoposets and Jauch–Piron states

In this section we will show the connection between the class of Boolean orthoposets and the class of orthoposets with a full (unital, resp.) set of two-valued Jauch–Piron states.

3.1. Theorem. *Every orthoposet with a full set of two-valued Jauch–Piron states is Boolean.*

Proof: Let P be an orthoposet with a full set of two-valued Jauch–Piron states and let $a, b \in P$ with $a \wedge b = 0$. Then there is no Jauch–Piron state s on P such that $s(a) = s(b) = 1$. Thus, $a \leq b'$ and $a \perp b$. The proof is complete.

There is also a partial converse to the previous theorem.

3.2. Definition. A non-zero element a of P is called *atom* if there is no $b \in P \setminus \{0, a\}$ with $b \leq a$.

3.3. Lemma. *Let P be a Boolean orthoposet and let $a \in P$ be an atom. Then the mapping $s : P \to \{0, 1\}$ defined by*

$$s(b) = \begin{cases} 0 & \text{if } a \not\leq b \\ 1 & \text{if } a \leq b \end{cases}$$

is a two-valued Jauch–Piron state on P.

Proof: It suffices to prove that for every $b \in P$ either $a \leq b$ or $a \perp b$. Let us suppose that $a \not\perp b$. Then there is a $c \in P \setminus \{0\}$ such that $c \leq a, b$. Since a is an atom, $c = a$ and $a \leq b$. The proof is complete.

3.4. Proposition. *Every atomic Boolean orthoposet has a full set of two-valued Jauch–Piron states.*

Proof: Let P be an atomic Boolean orthoposet and let $a, b \in P$ with $a \not\leq b$. Then there is a $c \in P \setminus \{0\}$ such that $c \leq a, b'$ and an atom $d \in P$ such that $d \leq c$. According to Lemma 3.3 there is a two-valued Jauch–Piron state s on P such that $s(a) = 1 \not\leq 0 = 1 - s(b') = s(b)$. The proof is complete.

According to Proposition 3.4, a Boolean orthoposet without a full set of two-valued Jauch–Piron states cannot be atomic. According to Theorem 2.2, it cannot have a unital set of two-valued Jauch–Piron states. We now give an example of a Boolean orthomodular poset without any two-valued Jauch–Piron state at all (Theorem 3.9). This answers a question posed in [3].

3.5. Example. Let S_0 be the square (in the plane $\mathbf{R}^2$) on Fig. 1. Let $\mathcal{P}_0 \subset \exp S_0$ be the concrete orthomodular poset generated by the family of polygons $G \subset S_0$ fulfilling the following conditions:

(1) Every interior angle of G is a multiple of $2\pi/8$.

(2) Let us denote by $A(x, \alpha)$ the open angle with the vertex in x such that the initial boundary halfline has zero angular coefficient and such that the terminal boundary halfline has the angular coefficient equal to α (anticlockwise) — see Fig. 2. A point x of the boundary of G belongs to G if and only if there is a disc D with the center in x and a number $\alpha > 0$ such that $D \cap A(x, \alpha) \subset P$.

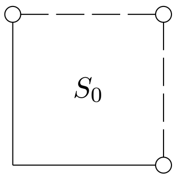

Fig. 1

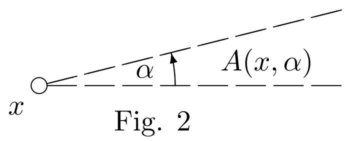

Fig. 2

It is easy to see that S_0 fulfils conditions (1) and (2). Other examples are given in Fig. 3. Let us observe that a homothetic image of a polygon fulfilling conditions (1) and (2) fulfils these conditions, too.

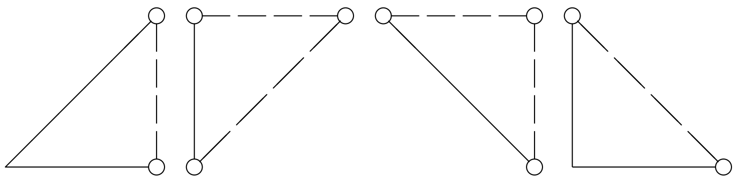

Fig. 3

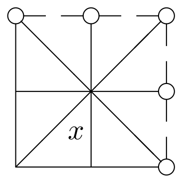

Fig. 4

3.6. Lemma. *$\mathcal{P}_0$ is a Boolean orthomodular poset.*

Proof: Let $A, B \in \mathcal{P}_0$ with $A \cap B \neq \emptyset$. Then there are an $x \in S_0$, open discs D_1, D_2 with centers in x and numbers $\alpha_1, \alpha_2 > 0$ such that $D_1 \cap A(x, \alpha_1) \subset A$, $D_2 \cap A(x, \alpha_2) \subset B$. Hence, $A \cap B$ contains a nonempty open subset $(D_1 \cap D_2) \cap A(x, \min(\alpha_1, \alpha_2))$. Thus, there is a $C \in \mathcal{P}_0$ with $C \subset A \cap B$ (e.g. homothetic to S_0).

3.7. Lemma. *Every element $A \in \mathcal{P}_0$ satisfies the following condition: For every $x \in \mathbf{R}^2$ either* $\operatorname{dist}(x, A) > 0$ *or there is a positive integer k and an $\varepsilon > 0$ such that $\lambda(A \cap D)/\lambda(D) = k/8$ for every disc D with the center in x and the radius less than ε (λ denotes the two-dimensional Lebesgue measure).*

Proof: According to the definition of $\mathcal{P}_0$, the above condition is fulfilled for all generators of $\mathcal{P}_0$. It is easy to see that it is fulfilled for S_0, too. Since this condition is fulfilled for $A \cup B$ and for $S_0 \setminus A$ whenever $A, B \in \mathcal{P}_0$ are disjoint elements fulfilling it, it is fulfilled for every $A \in \mathcal{P}_0$.

3.8. Lemma. *There is no two-valued Jauch–Piron state on $\mathcal{P}_0$.*

Proof: Let s be a two-valued state on $\mathcal{P}_0$. Let $\mathcal{S}_n$ $(n \geq 1)$ be the covering of S_0 by means of 2^{n^2} squares from $\mathcal{P}_0$ with the area $\lambda(S_0)/2^{n^2}$. Then there is an $S_n \in \mathcal{S}_n$ such that $s(S_n) = 1$. Since $S_{n+1} \subset S_n$ for every natural number n, there is an $x \in \mathbf{R}^2$ such that x belongs to the closure of all S_n. Thus, $s(A) = 0$ for every $A \in \mathcal{P}_0$ with $\operatorname{dist}(x, A) > 0$.

Now, let us take two finite systems $\mathcal{F}, \mathcal{F}' \subset \mathcal{P}_0$ of mutually disjoint elements such that the following conditions hold:

(1) $\operatorname{dist}(x, S_0 \setminus \bigcup \mathcal{F}) > 0$, $\operatorname{dist}(x, S_0 \setminus \bigcup \mathcal{F}') > 0$.

(2) There is an $\varepsilon > 0$ such that $\lambda(B \cap B' \cap D)/\lambda(D) = 1/16$ for every $B \in \mathcal{F}$, $B' \in \mathcal{F}'$ and for every disc D with the center in x and the radius less than ε.

Indeed, we can cover the neighbourhood of x (in S_0) by 8 (4 or 2 if x is on the boundary of S_0) triangles from $\mathcal{P}_0$ (see Fig. 4) and obtain thus $\mathcal{F}$. $\mathcal{F}'$ we can obtain by rotating the whole situation around x by $-2\pi/16$. (If x is on the boundary of S_0, one triangle will not be a subset of S_0 — instead of it we take the orthocomplement of the union of remaining ones.)

According to the condition (1), there are $B \in \mathcal{F}$, $B' \in \mathcal{F}'$ such that $s(B) = s(B') = 1$. According to the Lemma 3.7 and the condition (2), $s(A) = 0$ for every $A \in \mathcal{P}_0$ with $A \subset B \cap B'$. Thus, s is not Jauch–Piron. The proof is complete.

3.9. Theorem. *There is a Boolean orthomodular poset that has no two-valued Jauch–Piron state.*

Proof: It follows from Lemma 3.6 and Lemma 3.8.

3.10. Remark. (1) We can take another quotient than 1/8 in the definition of $\mathcal{P}_0$ and start with other suitable element than S_0.

(2) We can fix some directions of boundaries of elements of $\mathcal{P}_0$ and obtain thus different examples. For instance, we can allow only lines with the angular coefficient equal to a multiple of $2\pi/16$.

(3) It can be shown that the orthomodular poset generated by open balls in the space $\mathbf{R}^n$ $(n \geq 3)$ is a Boolean orthoposet with only one two-valued Jauch–Piron state that takes the value 0 exactly on bounded sets. (The author conjectures that this statement is valid also for $n = 2$, but this case brings about rather high combinatorial problems.)

Let $\mathcal{B}$ be the class of Boolean orthoposets, $\mathcal{F}$ ($\mathcal{U}$, resp.) be the class of orthoposets with a full (unital, resp.) set of two-valued Jauch–Piron states. We have shown (Theorems 3.1, 2.2 and 3.9) that $\mathcal{B} \cap \mathcal{U} = \mathcal{F}$ and that $\mathcal{B} \setminus \mathcal{U} \neq \emptyset$. A natural question arises whether $\mathcal{U} \setminus \mathcal{B} \neq \emptyset$. As the following example (it is a modification of an example from [4]) shows, the answer to that question is positive.

3.11. Example. Let $\{a\}$, $\{b\}$, X, Y be mutually disjoint sets such that X and Y are infinite. Let

$$\begin{aligned} \mathcal{P}' &= \{\emptyset,\ \{a\} \cup X,\ \{a\} \cup Y,\ \{b\} \cup X,\ \{b\} \cup Y,\ \{a, b\} \cup X \cup Y\}, \\ \mathcal{P} &= \{B;\ (B \setminus A) \cup (A \setminus B) \text{ is a finite subset of } X \cup Y \text{ for some } A \in \mathcal{P}'\}. \end{aligned}$$

Then $\mathcal{P}$ is a concrete orthomodular poset with a unital set of two-valued Jauch–Piron states that is not Boolean.

Proof: It is easy to see that $\mathcal{P}$ is a concrete orthomodular poset. Since $(\{a\} \cup X) \not\perp (\{a\} \cup Y)$ and $(\{a\} \cup X) \wedge (\{a\} \cup Y) = \emptyset$, $\mathcal{P}$ is not Boolean. Let us now for every $x \in X \cup Y$ define the two-valued state

$$s_x(A) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

for every $A \in \mathcal{P}$. Then the set $\{s_x;\ x \in X \cup Y\}$ is a unital set of two-valued Jauch–Piron states on $\mathcal{P}$.

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