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BOOLEAN ORTHOPOSETS—CONCRETENESS AND ORTHOCOMPLETENESS

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Summary. In this paper it is shown that a Boolean orthoposet (i.e., the orthoposet fulfilling the condition $a \wedge b \Rightarrow a \perp b$) admits a set representation. It is further shown that some results about Boolean orthoposets follow immediately from this representation. Finally, it is proved that an orthocomplete Boolean orthoposet has to be a Boolean algebra. This statement can be viewed as a generalization of various results from [3, 8, 5, 4].

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1. Preliminaries and basic notions

Let us first review basic notions.

Definition 1.1. An *orthoposet* is a triple $(P, \leq, ')$ such that
(1) $(P, \leq)$ is a partially ordered set with a least and a greatest elements 0, 1,
(2) $'\colon P \to P$ is an *orthocomplementation*, i.e., (i) $a'' = a$, (ii) $a \leq b \Rightarrow b' \leq a'$, (iii) $a \vee a' = 1$ for every $a, b \in P$.

In the sequel we will shortly write P instead of $(P, \leq, ')$.

Definition 1.2. Elements a, b of an orthoposet P are called *orthogonal* (denoted by $a \perp b$) if $a \leq b'$.
An orthoposet P is called *Boolean* if $a \perp b$ whenever $a \wedge b = 0$.

A special kind of orthoposets are the so-called concrete orthoposets—set represented orthoposets ordered by inclusion such that finite orthogonal suprema are formed by the set-theoretic unions.

Definition 1.3. A *concrete orthoposet* is a triple $(\mathcal{P}, \subset, ^c)$ where $\mathcal{P} \subset \exp X$ for some $X \neq \emptyset$ such that
(1) $\emptyset \in \mathcal{P}$,
(2) $A^c = X \setminus A \in \mathcal{P}$ whenever $A \in \mathcal{P}$,
(3) $\bigcup \mathcal{F} \in \mathcal{P}$ for every finite family $\mathcal{F} \subset \mathcal{P}$ of mutually disjoint elements such that $\bigvee \mathcal{F}$ exists in $(\mathcal{P}, \subset)$.

Definition 1.4. Orthoposets P, Q are called *isomorphic* if there is a one-to-one mapping $f\colon P \to Q$ such that
(1) $a \leq b$ if and only if $f(a) \leq f(b)$,
(2) $f(a') = f(a)'$
for every $a, b \in P$. The mapping f is then called an *isomorphism*.

2. Concrete representation of Boolean orthoposets

First, let us give a characterization of orthoposets admitting a concrete representation (this generalizes a result of [2]).

Definition 2.1. A *two-valued state* on an orthoposet $(P, \leq, ')$ is a mapping $s\colon P \to \{0, 1\}$ such that
(1) $s(1) = 1$,
(2) $s(a) \leq s(b)$ whenever $a \leq b$,
(3) $s(\bigvee F) = \sum_{a \in F} s(a)$ for every finite set $F \in P$ of mutually orthogonal elements such that $\bigvee F$ exists in $(P, \leq)$.

Definition 2.2. A set S of (not necessarily all) two-valued states on an orthoposet P is called *full* (see [2]) if for every pair $a, b \in P$ with $a \not\leq b$ there is a two-valued state $s \in S$ such that $s(a) \not\leq s(b)$.

Proposition 2.3. *An orthoposet P has a concrete representation iff the set of two-valued states on P is full.*

Proof. Let $f\colon P \to \mathcal{P}$ be an isomorphism of $(P, \leq, ')$ and $(\mathcal{P}, \subset, ^c)$ and let us suppose that $a, b \in P$ with $a \not\leq b$. Then $f(a) \not\subset f(b)$, hence there is an $x \in f(b) \setminus f(a)$. For the two-valued state s_x on P defined by

$$s_x(c) = 1 \quad \text{iff} \quad x \in f(c), \qquad c \in P,$$

we obtain $s_x(a) = 1 \not\leq 0 = s_x(b)$. Thus, the set of two-valued states on P is full. On the other hand, let us suppose that the set S of two-valued states on P is full. Then the mapping $f\colon P \to \exp S$ defined by

$$f(a) = \{s \in S;\ s(a) = 1\}$$

is an isomorphism of P and $f(P)$, where $f(P)$ is a concrete orthoposet. □

As a technical tool for constructing a concrete representation of a Boolean orthoposet we will use a special kind of ideals ([1], compare also with [6]).

Definition 2.4. Let P be an orthoposet. Let us define, for every set $Q \subset P$,

$$\begin{aligned} Q_+ &= \{a \in P;\ q \leq a \text{ for every } q \in Q\}, \\ Q_- &= \{a \in P;\ a \leq q \text{ for every } q \in Q\}. \end{aligned}$$

The set $I \subsetneqq P$ is called an *ideal* if $F_{+-} \subset I$ for every finite set $F \subset I$.
An ideal I is called *maximal* if it is not a proper subset of another ideal.

Lemma 2.5. *Let P be a Boolean orthoposet and let $I \subset P$ be an ideal. Let us suppose that $I \cap \{a, a'\} = \emptyset$ for some $a \in P$. Then $I \cup \{a\}$ is contained in an ideal.*

P r o o f. Put

$$J = \bigcup_{\substack{F \subset I \\ F \text{ finite}}} (F \cup \{a\})_{+-}\ .$$

Then $I \cup \{a\} \subset J$ and it suffices to prove that J is an ideal. First, let us suppose that $G \subset J$ is a finite set. Every $g \in G$ belongs to $(F_g \cup \{a\})_{+-}$ for some finite set $F_g \subset I$. Hence g is a lower bound for $(F_g \cup \{a\})_+$ and for $(\bigcup_{g \in G} F_g \cup \{a\})_+$. Since g is an arbitrary element of G, we obtain $G_+ \supset (\bigcup_{g \in G} F_g \cup \{a\})_+$ and $G_{+-} \subset (\bigcup_{g \in G} F_g \cup \{a\})_{+-} \subset J$. Now, let us suppose that $F \subset I$ is a finite set. Since $a' \notin F_{+-} \subset I$, there is a $b_F \in F_+$ such that $a' \not\leq b_F$. Since P is Boolean, there is a $c_F \in P \setminus \{0\}$ such that $c_F \leq a', b'_F$. Thus, $c'_F \neq 1$ is an upper bound for $F \cup \{a\}$ and $1 \notin (F \cup \{a\})_{+-}$. Therefore $J \neq P$. The proof is complete. □

Theorem 2.6. *Every Boolean orthoposet has a full set of two-valued states, i.e., it has a concrete representation.*

P r o o f. Let P be a Boolean orthoposet and let us suppose that $a, b \in P$ with $a \not\leq b$. Then $\{b\}_-$ is an ideal and, making use of Lemma 2.5 if necessary, we conclude that $\{b\}_- \cup \{a'\}$ is contained in an ideal. According to Zorn's lemma, there is a maximal ideal $I \supset \{b\} \cup \{a'\}$. Hence, according to Lemma 2.5, for every $c \in P$ either $c \in P$ or $c' \in P$. Since I is closed under (finite) orthogonal suprema that exist, it is easy to see that the mapping $s\colon P \to \{0,1\}$ defined by

$$s(c) = 0 \qquad \text{if and only if} \qquad c \in I$$

for every $c \in P$, is a two-valued state on P such that $s(a) = 1 \not\leq 0 = s(b)$. The proof is complete. □

3. Orthocompleteness in Boolean orthoposets

In this section we will show how some completeness conditions force a Boolean orthoposet to have "good" behavior. We will use Theorem 2.6. It should be noted that an alternative proof (using some distributivity property of Boolean orthoposets) of Corollary 3.4 and Proposition 3.5 is presented in [8].

Definition 3.1. Let α be a cardinal number. An orthoposet P is called α-*orthocomplete* if every set of cardinality less than α consisting of mutually orthogonal elements of P has a supremum.

An orthoposet is called *orthocomplete* if it is α-orthocomplete for every cardinal number α.

Definition 3.2. An orthoposet P is called *orthomodular* if for every pair $a, b \in P$ with $a \leq b$ there is a $c \in P$ such that $c \perp a$ and $b = a \vee c$.

Proposition 3.3. *Every ω_0-orthocomplete (ω_0 denotes the first infinite cardinal) concrete orthoposet is orthomodular.*

Proof. Let $\mathcal{P}$ be a concrete orthoposet and let $A, B \in \mathcal{P}$ with $A \subset B$. Then $B \setminus A = (A \cup B^c)^c \in \mathcal{P}$, $(B \setminus A) \perp A$ and $B = A \vee (B \setminus A)$. □

Corollary 3.4. *Every ω_0-orthocomplete Boolean orthoposet is orthomodular.*

Proposition 3.5. *Every lattice Boolean orthoposet is a Boolean algebra.*

Proof. Let $\mathcal{P}$ be a lattice concrete Boolean orthoposet and let $A, B \in \mathcal{P}$. Since $(A \setminus (A \wedge B)) \wedge (B \setminus (A \wedge B)) = \emptyset$, we have $(A \setminus (A \wedge B)) \cap (B \setminus (A \wedge B)) = \emptyset$, hence $A \cap B = A \wedge B \in \mathcal{P}$. Moreover, $A \cup B = (A^c \cap B^c)^c = (A^c \wedge B^c)^c = A \vee B \in \mathcal{P}$. Thus, $A \wedge (B \vee C) = A \cap (B \cup C) = (A \cap C) \cup (A \cap C) = (A \wedge C) \vee (A \wedge C)$ for every $A, B, C \in \mathcal{P}$. The proof is complete. □

The following theorem is a generalization of various results from [3, 8, 5, 4].

Theorem 3.6. *Every orthocomplete Boolean orthoposet is a Boolean algebra.*

P r o o f. Let us suppose that P is an orthocomplete Boolean orthoposet. According to Proposition 3.5, it suffices to prove that $a \wedge b$ exists for every $a, b \in P$. Let $a, b \in P$. According to Zorn's lemma, there is a nonempty maximal set $Q \subset P$ of mutually orthogonal elements $c \leq a, b$. Since P is orthocomplete, there is a $q \in P$ such that $q = \bigvee Q$. Let us suppose that $q \neq a \wedge b$ and seek a contradiction. There is a $d \leq a, b$ such that $d \not\leq q$. Since P is Boolean, there is an $e \in P \setminus \{0\}$ such that $e \leq d, q'$. Thus, the set $Q \cup \{e\}$ consists of mutually orthogonal elements that are less than or equal to a, b, which contradicts the maximality of Q. □

Let us note that an orthocomplete Boolean algebra is complete. As the following proposition shows, the condition of orthocompleteness in Theorem 3.6 cannot be weakened to α-orthocompleteness for any cardinal number α.

Proposition 3.7. *For every cardinal number α there is an α-complete orthomodular Boolean orthoposet that is not a Boolean algebra.*

P r o o f. Without any loss of generality we may (and will) suppose that α is infinite. Let X_1, X_2, X_3, X_4 be mutually disjoint sets of cardinality α^+. Let us denote $S \triangle T = (S \setminus T) \cup (T \setminus S)$ for every pair S, T of sets and let us put

$$\begin{aligned} X &= X_1 \cup X_2 \cup X_3 \cup X_4, \\ \mathcal{P}' &= \{\emptyset,\ X_1 \cup X_2,\ X_2 \cup X_3,\ X_3 \cup X_4,\ X_4 \cup X_1,\ X\}, \\ \mathcal{P} &= \{A \triangle B;\ A \in \mathcal{P}',\ B \subset X \text{ and } \operatorname{card} B \leq \alpha\}. \end{aligned}$$

It is only a routine verification that $\mathcal{P}$ is a concrete α^+-complete orthoposet. Indeed, $\emptyset \in \mathcal{P}$ and $(A \triangle B)^c = A^c \triangle B \in \mathcal{P}$ for every $A \triangle B \in \mathcal{P}$ (of the form in the definition of $\mathcal{P}$). Finally, let $A_\beta \triangle B_\beta$ $(\beta \in \alpha)$ be mutually orthogonal elements of $\mathcal{P}$ (of the form in the definition of $\mathcal{P}$). Then $\bigcup_{\beta \in \alpha}(A_\beta \triangle B_\beta) = (\bigcup_{\beta \in \alpha} A_\beta) \triangle C$ for some $C \subset X$ with $\operatorname{card} C \leq \sum_{\beta \in \alpha} \operatorname{card} B_\beta \leq \alpha$. Since A_β $(\beta \in \alpha)$ are mutually orthogonal sets, we have $\bigcup_{\beta \in \alpha} A_\beta \in \mathcal{P}'$ and $\bigvee_{\beta \in \alpha}(A_\beta \triangle B_\beta) = \bigcup_{\beta \in \alpha}(A_\beta \triangle B_\beta) \in \mathcal{P}$.

According to Corollary 3.4, $\mathcal{P}$ is orthomodular. Since $\{x\} \in \mathcal{P}$ for every $x \in X$, $\mathcal{P}$ is Boolean. It suffices to prove that $\mathcal{P}$ is not a Boolean algebra. Indeed, $(X_1 \cup X_2), (X_2 \cup X_3) \in \mathcal{P}$ and $\{A \in \mathcal{P};\ A \subset X_1 \cup X_2 \text{ and } A \subset X_2 \cup X_3\} = \{A \in \mathcal{P};\ A \subset X_2 \text{ and } \operatorname{card} A \leq \alpha\}$ does not have a maximal element, i.e., $(X_1 \cup X_2) \wedge (X_2 \cup X_3)$ does not exist. □

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