Commutative Bounded Integral Residuated Orthomodular Lattices are Boolean Algebras

Josef Tkadlec* Esko Turunen†

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Abstract We show that a commutative bounded integral orthomodular lattice is residuated iff it is a Boolean algebra. This result is a consequence of [7, Theorem 7.31]; however, our proof is independent and uses other instruments.

Keywords Residuated lattice · Orthomodular lattice

1 Commutative Bounded Integral Residuated Orthomodular Lattices

Residuated lattices were first studied by Dilworth [1] in 1938. Recently they have became important in many-valued logic framework. Indeed, Hájek's *BL*-algebras, Chang's *MV*-algebras and Girard monoids—they rise as Lindenbaum algebras from certain logical axioms in a similar manner than Boolean algebras do from Classical logic—are specific cases of residuated lattices as they are commutative, bounded, integral residuated lattices. More precisely, a lattice $L = \langle L, \leq, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ with the least element $\mathbf{0}$ and the largest element $\mathbf{1}$ is called *commutative, bounded, integral residuated lattice* if it is endowed with a couple of binary operations $\langle \odot, \rightarrow\rangle$ (called *adjoint couple*) such that $\odot$ is associative, commutative, isotone and $x \odot \mathbf{1} = x$ holds for all elements $x \in L$. Hence for every $x, y \in L$ we obtain

$$x \odot y \leq (x \odot \mathbf{1}) \wedge (\mathbf{1} \odot y) \leq x \wedge y\,.$$

Moreover, a *Galois connection*

$$x \odot y \leq z \quad \text{iff} \quad x \leq y \rightarrow z$$

holds for all elements $x, y, z \in L$, for detail, see e.g. [2, 3, 6]. In fact, there is a little bit of variations in terminology, Höhle [3] for example, calls such structures commutative, residuated, integral ℓ–monoids.

Notice that, in particular, the meet operation $\wedge$ is associative, commutative, isotone and $x \wedge \mathbf{1} = x$ holds for all elements $x \in L$. Thus, it is relevant to study lattices that can be considered as residuated with an adjoint couple $\langle \wedge, \rightarrow\rangle$. It is well-known that Boolean algebras are such lattices. In these algebraic structures the *residuum* operation $\rightarrow$ is defined by a stipulation $x \rightarrow y = \neg x \vee y$.

The unit real interval $[0, 1]$, too, can be considered as a residuated lattice where

$$x \wedge y = \min\{x, y\}\,, \qquad x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y\,, \\ y & \text{otherwise.} \end{cases}$$

This structure is called *Gödel algebra*, obviously it is commutative bounded and integral. Notice that the Lindenbaum algebra of the corresponding *Gödel logic* is another example of a (commutative bounded integral) residuated lattice with an adjoint couple $\langle \wedge, \rightarrow\rangle$.

Orthomodular lattices (or more generally orthomodular posets) are studied as quantum logics, see e.g. [4]. An *ortholattice* is a lattice $\langle L, \leq, \wedge, \vee, ', \mathbf{0}, \mathbf{1}\rangle$ with the least element $\mathbf{0}$, the greatest element $\mathbf{1}$ and the *orthocomplementation* $' \colon L \rightarrow L$ fulfilling the properties (a) $x'' = x$ for every $x \in L$, (b) $x \leq y$ implies $y' \leq x'$ for every $x, y \in L$, (c) $x \vee x' = \mathbf{1}$ for every $x \in L$. An *orthomodular lattice* is an ortholattice L fulfilling the *orthomodular law*: $y = x \vee (x' \wedge y)$ for every $x, y \in L$ with $x \leq y$.

The main motivation to write this paper is to specify such a lattice structure that would be interesting both in many–valued logics framework and in quantum logics framework, thus a commutative, bounded, integral, residuated orthomodular lattice. It turns out, however, that the only such lattices are Boolean algebras which are uninteresting both in many–valued logics and in quantum logics framework. In fact, our result is not new, if follows from Theorem 7.31 in [7] stating that the only complemented lattices which can be residuated are Boolean algebras. However, our proof is independent. Moreover, we assume that this negative

*Czech Technical University in Prague, Faculty of Electrical Engineering, 166 27 Praha, Czech Republic

†Tampere University of Technology, P.O. Box 553, 33101 Tampere, Finland, e-mail: esko,tutunen@tut.fi

result in generally unknown in both many–valued logics community and quantum logics community.

We will use a characterization of Boolean algebras in orthomodular lattices that is a consequence of the characterization of Boolean algebras in orthomodular posets given by Tkadlec [5], however we will present its proof here.

Let us review some notions and properties of orthomodular lattices. Elements x, y of an orthomodular lattice are called *orthogonal* (denoted by $x \perp y$) if $x \leq y'$. Let us denote $y - x = y \wedge x'$ for $x \leq y$. Then for every $x \leq y$ we have $x \perp (y - x)$ and, according to the orthomodular law, $y = x \vee (y - x)$. For every pair x, y of elements of an orthomodular lattice we have and $\big(x - (x \wedge y)\big) \wedge y = \big(x \wedge (x \wedge y)'\big) \wedge y = (x \wedge y) \wedge (x \wedge y)' = \big((x \wedge y)' \vee (x \wedge y)\big)' = \mathbf{1}' = \mathbf{0}$. It is well-known (see e.g. [4]) that an orthomodular lattice is a Boolean algebra iff every pair x, y of its elements is compatible, i.e., $x - (x \wedge y)$ and $y - (x \wedge y)$ are orthogonal.

Theorem 1 *An orthomodular lattice* $\langle L, \leq, \wedge, \vee, ', \mathbf{0}, \mathbf{1}\rangle$ *is a Boolean algebra iff for every* $x, y \in L$ *the condition* $x \wedge y = x \wedge y' = \mathbf{0}$ *implies* $x = \mathbf{0}$.

Proof $\Rightarrow$: Let $x, y \in L$ be such that $x \wedge y = x \wedge y' = \mathbf{0}$. Using the distributivity we obtain that $x = x \wedge \mathbf{1} = x \wedge (y \vee y') = (x \wedge y) \vee (x \wedge y') = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$.

$\Leftarrow$: We will show that every pair $x, y \in L$ is compatible. Let us denote $z = x \wedge y$, $u = (x - z) \wedge y'$. Then $\big((x - z) - u\big) \wedge y' = \mathbf{0}$ and, since $(x - z) \wedge y = \mathbf{0}$, also $\big((x - z) - u\big) \wedge y = \mathbf{0}$. According to the assumption, $(x - z) - u = \mathbf{0}$ and therefore $x - z = u$. Since $u \perp y$, we have $u \perp (y - z)$ and therefore $x - (x \wedge y)$ and $y - (x \wedge y)$ are orthogonal. □

Using this characterization we can prove the main result of this paper.

Theorem 2 *An orthomodular lattice is residuated iff it is a Boolean algebra.*

Proof $\Leftarrow$: As we have already mentioned, a Boolean algebra is residuated with $\langle \wedge, \to \rangle$ for the residuum operation $\to$ defined by $x \to y = \neg x \vee y$.

$\Rightarrow$: It suffices to check the condition in Theorem 1. Let us suppose that $x \wedge y = x \wedge y' = \mathbf{0}$ for elements x, y of the lattice in question. Since $x \odot y \leq x \wedge y$ and $x \odot y' \leq x \wedge y'$, we obtain $x \odot y \leq \mathbf{0}$ and $x \odot y' \leq \mathbf{0}$. The Galois connection gives $y \leq (x \to \mathbf{0})$ and $y' \leq (x \to \mathbf{0})$. Since $\mathbf{1} = y \vee y'$, we obtain $\mathbf{1} \leq (x \to \mathbf{0})$. The Galois connection gives $\mathbf{1} \odot x \leq \mathbf{0}$. Since $\mathbf{1} \odot x = x$ and $\mathbf{0}$ is the least element of the lattice, we obtain $x = \mathbf{0}$. The proof is complete. □

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References

[1] Dilworth, R. P.: Abstract residuation over lattices, Bull. Amer. Math. Soc. **44**, 262–268 (1938).

[2] Hájek, P.: Metamathematics of Fuzzy Logic, Kluwer (1998).

[3] Höhle, U.: On the Fundamentals of Fuzzy Set Theory, J. Math. Anal. Applications **201**, 786–826 (1996).

[4] Pták, P., Pulmannová, S.: Orthomodular Structures as Quantum Logics. Kluwer Academic Publishers, (1991).

[5] Tkadlec, J.: Conditions that force an orthomodular poset to be a Boolean algebra. Tatra Mt. Math. Publ. **10**, 55–62 (1997).

[6] Turunen, E.: Mathematics behind Fuzzy Logic. Springer-Verlag, New York (1999).

[7] Ward, M., Dilworth R. P.: Residuated Lattices, Trans. Amer. Math. Soc. **45**, 335–354 (1939).