

Effect algebras with the maximality property

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Abstract

The maximality property was introduced in [9] in orthomodular posets as a common generalization of orthomodular lattices and orthocomplete orthomodular posets. We show that various conditions used in the theory of effect algebras are stronger than the maximality property, clear up the connections between them and show some consequences of these conditions. In particular, we prove that a Jauch–Piron effect algebra with a countable unital set of states is an orthomodular lattice and that a unital set of Jauch–Piron states on an effect algebra with the maximality property is strongly order determining.

1 Basic notions

Effect algebras as generalizations of orthomodular posets (quantum logics) are studied in the axiomatics of quantum systems—see, e.g., [1, 2].

1.1. Definition. An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are distinct elements of E and \oplus is a partial binary operation on E such that for every $a, b, c \in E$ the following conditions hold:

- (1) $a \oplus b = b \oplus a$ if $a \oplus b$ exists,
- (2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if $(a \oplus b) \oplus c$ exists,
- (3) there is a unique $a' \in E$ such that $a \oplus a' = \mathbf{1}$ (*orthosupplement*),
- (4) $a = \mathbf{0}$ whenever $a \oplus \mathbf{1}$ is defined.

For simplicity, we use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \leq b$ iff there is a $c \in E$ such that $b = a \oplus c$. Such an element c is unique (if it exists) and is denoted by $b \ominus a$. The element $\mathbf{0}$ ($\mathbf{1}$, resp.) is the least (the greatest, resp.) element of E with respect to this partial ordering. For every $a, b \in E$, $a'' = a$ and $b' \leq a'$ whenever $a \leq b$. It can be shown that $a \oplus \mathbf{0} = a$ for every $a \in E$ and that a *cancellation law* is valid: for every $a, b, c \in E$ with $a \oplus b \leq a \oplus c$ we have $b \leq c$. An *orthogonality* relation on E is defined by $a \perp b$ iff $a \oplus b$ exists (iff $a \leq b'$). (See, e.g., [1, 2].)

For $a \leq b$ we denote $[a, b] = \{c \in E : a \leq c \leq b\}$. A *chain* in E is a nonempty linearly (totally) ordered subset of E .

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Obviously, if $a \perp b$ and $a \vee b$ exist in an effect algebra, then $a \vee b \leq a \oplus b$. The reverse inequality need not be true (it holds in orthomodular posets).

1.2. Definition. Let E be an effect algebra. An element $a \in E$ is *principal* if $b \oplus c \leq a$ for every $b, c \in E$ such that $b, c \leq a$ and $b \perp c$.

1.3. Definition. An *orthoalgebra* is an effect algebra E in which, for every $a \in E$, $a = \mathbf{0}$ whenever $a \oplus a$ is defined.

An *orthomodular poset* is an effect algebra in which every element is principal.

An *orthomodular lattice* is an orthomodular poset that is a lattice.

Every orthomodular poset is an orthoalgebra. Indeed, if $a \oplus a$ is defined then $a \oplus a \leq a = a \oplus \mathbf{0}$ and, according to the cancellation law, $a \leq \mathbf{0}$ and therefore $a = \mathbf{0}$.

Orthoalgebras are characterized by the following conditions: the orthosupplementation is an orthocomplementation (i.e., $a \vee a' = \mathbf{1}$ for every a) or $a \oplus b$ is a minimal upper bound of a, b for every a, b . Orthomodular posets are characterized as effect algebras such that $a \oplus b = a \vee b$ for every orthogonal pair a, b . (See [2, 3].) Let us remark that an orthomodular poset is usually defined as a bounded partially ordered set with an orthocomplementation in which the orthomodular law is valid.

Let us present a special construction of orthomodular posets that we will use in some examples.

1.4. Proposition. Let $X \neq \emptyset$, $E \subseteq \exp X$ be nonempty such that the following conditions are fulfilled:

- (1) $X \setminus A \in E$ whenever $A \in E$,
- (2) $A \cup B \in E$ whenever $A, B \in E$ are disjoint.

Then $(E, \oplus, \emptyset, X)$ with $A \oplus B = A \cup B$ for disjoint $A, B \in E$ is an orthomodular poset such that the orthosupplement is the set-theoretic complement and the partial ordering is the inclusion.

Proof. Obvious.

1.5. Definition. An orthomodular poset of the form of Proposition 1.4 is called *concrete*.

It is well-known (see, e.g., [8]) that an orthomodular poset has a concrete representation iff it has a strongly order determining set of two-valued states (see the next section for these notions).

Let us present two important notions we will use in the sequel.

1.6. Definition. A system $(a_i)_{i \in I}$ of (not necessarily distinct) elements of an effect algebra E is *orthogonal* if $\bigoplus_{i \in F} a_i$ is defined for every finite set $F \subseteq I$.

An effect algebra E is *orthocomplete* if for every orthogonal system $(a_i)_{i \in I}$ of elements of E the supremum $\bigvee \{\bigoplus_{i \in F} a_i : F \subseteq I \text{ is finite}\}$ exists.

1.7. Definition. An effect algebra E has the *maximality property* if $[\mathbf{0}, a] \cap [\mathbf{0}, b]$ has a maximal element for every $a, b \in E$.

Obviously, every finite effect algebra has the maximality property and every lattice effect algebra has the maximality property: $a \wedge b$ is a maximal (even the greatest) element of $[\mathbf{0}, a] \cap [\mathbf{0}, b]$ for every a, b .

2 States

2.1. Definition. Let E be an effect algebra. A *state* s on E is a mapping $s: E \rightarrow [0, 1]$ such that:

- (1) $s(\mathbf{1}) = 1$,
- (2) $s(a \oplus b) = s(a) + s(b)$ whenever $a \oplus b$ is defined.

A set S of states on E is *unital* if, for every $a \in E \setminus \{\mathbf{0}\}$, there is a state $s \in S$ such that $s(a) = 1$.

A set S of states on E is *strongly order determining* if, for every $a, b \in E$ with $a \not\leq b$, there is a state $s \in S$ such that $s(a) = 1 > s(b)$.

Obviously, for every state s we have $s(\mathbf{0}) = 0$, $s(a') = 1 - s(a)$ for every $a \in E$, $s(a) \leq s(b)$ for every $a, b \in E$ with $a \leq b$. It is easy to see that every strongly order determining set of states on an effect algebra is unital.

There are orthomodular lattices without any state [4]. On the other hand, there is a strongly order determining set of special two-valued states on concrete orthomodular posets (it is easy to verify that they are indeed states):

2.2. Definition. Let $E \subseteq \exp X$ be a concrete orthomodular poset, $x \in X$. The state s_x on E defined by

$$s_x(A) = \begin{cases} 0, & x \notin A, \\ 1, & x \in A, \end{cases} \quad A \in E,$$

is called *carried by the point* x .

Let us present two observations describing the impact of a sufficiently large state spaces to the properties of the algebraic structure.

2.3. Proposition. *Every effect algebra with a unital set of states is an orthoalgebra.*

Proof. Let E be an effect algebra with a unital set S of states. Let $a \in E$ be such that $a \oplus a$ is defined. Then $1 \geq s(a \oplus a) = 2s(a)$ and therefore $s(a) \leq \frac{1}{2}$ for every state $s \in S$. Since S is unital, we obtain that $a = \mathbf{0}$.

2.4. Proposition. *Every effect algebra with a strongly order determining set of states is an orthomodular poset.*

Proof. Let E be an effect algebra with a strongly order determining set S of states. Let us prove that every element of E is principal. Let $a, b, c \in E$ be such that $b, c \leq a$ and $b \perp c$. Then for every state $s \in S$ with $s(a') = 1$ we consecutively obtain: $0 = s(a) = s(b) = s(c) = s(b \oplus c)$, $s((b \oplus c)') = 1$. Since the set S is strongly order determining, we obtain that $a' \leq (b \oplus c)'$ and therefore $b \oplus c \leq a$.

3 Jauch–Pironness

An important role in the axiomatics of quantum structures play the so-called Jauch–Piron states [5, 8].

3.1. Definition. Let E be an effect algebra. A state s on E is *Jauch–Piron* if for every $a, b \in E$ with $s(a) = s(b) = 1$ there is a $c \in E$ such that $c \leq a, b$ and $s(c) = 1$.

An effect algebra is *Jauch–Piron* if every state on it is Jauch–Piron.

It is easy to see that every Boolean algebra is Jauch–Piron. There are Jauch–Piron effect algebras that are not orthoalgebras (e.g., the 3-element chain $C_3 = \{\mathbf{0}, a, \mathbf{1}\}$ with $a \oplus a = \mathbf{1}$ and $x \oplus \mathbf{0} = x$ for every $x \in C_3$) and orthomodular lattices that are not Jauch–Piron (e.g., $\text{MO}_2 = \{\mathbf{0}, a, a', b, b', \mathbf{1}\}$ with $a \oplus a' = b \oplus b' = \mathbf{1}$ and $x \oplus \mathbf{0} = x$ for every $x \in E$; the two-valued states are not Jauch–Piron).

The following statement was proved in [10, Proposition 2.6], we will generalize it later (Theorem 3.7).

3.2. Lemma. *Every Jauch–Piron effect algebra with a countable unital set of states has the maximality property.*

3.3. Proposition. *Every effect algebra with the maximality property and with a unital set of Jauch–Piron states is an orthomodular poset.*

Proof. Let E be an effect algebra with the maximality property and with a unital set S of Jauch–Piron states. Let us suppose that E is not an orthomodular poset and seek a contradiction. There are elements $a, b, c \in E$ such that $b, c \leq a$, $b \perp c$ and $b \oplus c \not\leq a$. Let us denote $d = b \oplus c$. Since E has the maximality property, there is a maximal element e in $[\mathbf{0}, a'] \cap [\mathbf{0}, d']$. Since $d \not\leq a$, we obtain that $a' \not\leq d'$ and therefore $e < a'$ and $a' \ominus e \neq \mathbf{0}$. Since the set S is unital, there is a state $s \in S$ such that $s(a' \ominus e) = 1$. Hence $s(a') = 1$, $0 = s(e) = s(a) = s(b) = s(c) = s(d)$, $s(d') = 1$, $s(d' \ominus e) = 1$. Since the state s is Jauch–Piron, there is an element $f \in E$ such that $f \leq (a' \ominus e), (d' \ominus e)$ and $s(f) = 1$. Hence $f \neq \mathbf{0}$ and $e < e \oplus f \leq a', d'$ —this contradicts to the maximality of e .

Let us remark that there are effect algebras with the maximality property that are not orthoalgebras—e.g., the 3-element chain $C_3 = \{\mathbf{0}, a, \mathbf{1}\}$ with $a \oplus a = \mathbf{1}$ and $x \oplus \mathbf{0} = x$ for every $x \in C_3$. It seems to be an open question whether the assumption of the maximality property in Proposition 3.3 might be omitted (it is not a consequence of the existence of a countable unital set of Jauch–Piron states—see Example 3.8). Proposition 3.3 cannot be improved to orthomodular lattices—see Example 3.5 ($\{\frac{1}{2}(s_x + s_y) : x, y \in X, x \neq y\}$ is a unital set of Jauch–Piron states).

It is well-known and easy to see that every state on a Boolean algebra is Jauch–Piron and that a unital set of states on a Boolean algebra is strongly order determining. Let us generalize the latter statement.

3.4. Theorem. *A set of Jauch–Piron states on an effect algebra with the maximality property is unital if and only if it is strongly order determining.*

Proof. \Leftarrow : Obvious.

\Rightarrow : Let E be an effect algebra with the maximality property and with a unital set S of Jauch–Piron states. Let $a, b \in E$ such that $a \not\leq b$. Let $c \in E$ be a maximal element of $[0, a] \cap [0, b]$. Then $c < a$ and therefore $a \ominus c \neq 0$. Since the set S is unital, there is a state $s \in S$ such that $s(a \ominus c) = 1$ and therefore $s(a) = 1$. Let us suppose that $s(b) = 1$ and seek a contradiction. Since s is Jauch–Piron, there is an element $d \in E$ such that $d \leq a \ominus c$, $d \leq b$ and $s(d) = 1$. Hence $d \neq 0$ and $c < c \oplus d \leq a$. According to Proposition 3.3, b is principal and therefore $c \oplus d \leq b$ —this contradicts to the maximality of c .

Let us remark that we proved Theorem 3.4 using Proposition 3.3. On the other hand, Proposition 3.3 is a direct consequence of Theorem 3.4 and Proposition 2.4, hence we might prove it by this way if we prove Theorem 3.4 by another way.

Let us present examples showing that the assumptions in Theorem 3.4 cannot be omitted.

3.5. Example. Let $X = \{a, b, c, d\}$, E be the family of even-element subsets of X with the \oplus operation defined as the union of disjoint sets. Then $(E, \oplus, \emptyset, X)$ is a finite (hence with the maximality property) concrete orthomodular poset and the set $S = \{s_a, s_b, s_c\}$ of states carried by points a, b, c is a unital set of (two-valued) states on E that is not strongly order determining: $\{a, d\} \not\leq \{a, b\}$ but there is no state $s \in S$ such that $s(\{a, d\}) = 1 > s(\{a, b\})$. (States in S are not Jauch–Piron.)

3.6. Example. Let X_1, X_2, X_3, X_4 be nonempty mutually disjoint sets, X_1, X_3 be infinite, $X = \bigcup_{i=1}^4 X_i$,

$$E_0 = \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\},$$

$$E = \{(A \setminus F) \cup (F \setminus A) : F \subseteq X_1 \cup X_3 \text{ is finite}, A \in E_0\},$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X)$ is a concrete orthomodular poset and the set $S = \{s_x : x \in X_1 \cup X_3\}$ of states carried by points from $X_1 \cup X_3$ is a unital set of (two-valued) Jauch–Piron states on E . The set S is not strongly order determining because $X_1 \cup X_4 \not\leq X_1 \cup X_2$ and for every $s \in S$ with $s(X_1 \cup X_4) = 1$ there is an $x \in X_1$ such that $s = s_x$ and therefore $s(X_1 \cup X_2) = 1$. (E does not have the maximality property.)

3.7. Theorem. *Every Jauch–Piron effect algebra with a countable unital set of states is an orthomodular lattice.*

Proof. Let E be a Jauch–Piron effect algebra with a countable unital set S of states. According to Lemma 3.2, E has the maximality property. According to Theorem 3.4, the set S is strongly order determining. According to Proposition 2.4, E is an orthomodular poset. Let us show that $a \wedge b$ exists for every $a, b \in E$. (Then also $a \vee b = (a' \wedge b')'$ exists for every $a, b \in E$.) If $[0, a] \cap [0, b] = \{0\}$ then $0 = a \wedge b$. Let us suppose that there is a nonzero element $c \in E$ such that $c \leq a, b$. Then there is a state $s \in S$ such that $s(c) = 1$. Hence $s(a) = s(b) = 1$ and the set $S_{a,b} = \{s \in S : s(a) = s(b) = 1\}$ is nonempty and countable. Let s_0 be a σ -convex combination with nonzero coefficients of all (possibly infinitely many) states from $S_{a,b}$, i.e., there are real numbers $\alpha_s > 0$ for $s \in S_{a,b}$ such that $\sum_{s \in S_{a,b}} \alpha_s = 1$ and

$s_0 = \sum_{s \in S_{a,b}} \alpha_s s$. Then s_0 is a state on E such that $s_0(a) = s_0(b) = 1$. Since the state s_0 is Jauch–Piron, there is an element $c_0 \in E$ such that $c_0 \leq a, b$ and $1 = s_0(c_0) = \sum_{s \in S_{a,b}} \alpha_s s(c_0)$. Hence, for every $s \in S_{a,b}$, $\alpha_s s(c_0) = \alpha_s$ and therefore $s(c_0) = 1$. For every $c \in [\mathbf{0}, a] \cap [\mathbf{0}, b]$ and every $s \in S$ with $s(c) = 1$ we have $s \in S_{a,b}$ and therefore $s(c_0) = 1$. Since S is strongly order determining, $c \leq c_0$ for every $c \in [\mathbf{0}, a] \cap [\mathbf{0}, b]$. Hence $c_0 = a \wedge b$.

Let us present showing examples that the conditions in Theorem 3.7 cannot be omitted. There is a concrete (hence with a strongly order determining set of two-valued states) Jauch–Piron orthomodular poset that is not a lattice—see [7] (every unital set of states on it is uncountable). As the following example shows there is an orthomodular poset with a countable strongly order determining set of (two-valued) Jauch–Piron states that does not have the maximality property and therefore it is not a lattice (there are non-Jauch–Piron states).

3.8. Example. Let X_1, X_2, X_3, X_4 be mutually disjoint countable infinite sets, $X = \bigcup_{i=1}^4 X_i$,

$$\begin{aligned} E_0 &= \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\}, \\ E &= \{(A \setminus F) \cup (F \setminus A) : F \subseteq X \text{ is finite}, A \in E_0\}, \end{aligned}$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X)$ is a concrete orthomodular poset and the set $S = \{s_x : x \in X\}$ of states carried by points is a countable strongly order determining set of two-valued Jauch–Piron states on E . The set $[\emptyset, X_1 \cup X_2] \cap [\emptyset, X_4 \cup X_1]$ consists of finite subsets of X_1 , hence E does not have the maximality property. As an example of a non-Jauch–Piron state we can take a σ -convex combination with nonzero coefficients of states from $\{s_x : x \in X_1\}$.

4 Relationship of various conditions

4.1. Theorem. *Let E be an effect algebra. Consider the following properties:*

- (F) E is finite.
- (CF) E is chain-finite.
- (OC) E is orthocomplete.
- (JPCU) E is Jauch–Piron with a countable unital set of states.
- (L) E is a lattice.
- (CU) For every $a, b \in E$, every chain in $[\mathbf{0}, a] \cap [\mathbf{0}, b]$ has an upper bound in $[\mathbf{0}, a] \cap [\mathbf{0}, b]$.
- (M) E has the maximality property.

Then the following implications hold: (F) \Rightarrow (CF) \Rightarrow (OC) \Rightarrow (CU) \Rightarrow (M), (JPCU) \Rightarrow (L) \Rightarrow (CU).

Proof. (F) \Rightarrow (CF): Obvious.

(CF) \Rightarrow (OC): Every orthogonal system in a chain-finite effect algebra is finite. Hence E is orthocomplete.

(OC) \Rightarrow (CU): Let C be a chain in $[\mathbf{0}, a] \cap [\mathbf{0}, b]$. According to [6, Theorem 3.2], every chain in an orthocomplete effect algebra has a supremum. This supremum obviously belongs to $[\mathbf{0}, a] \cap [\mathbf{0}, b]$.

(CU) \Rightarrow (M): Let $a, b \in E$. Since $[\mathbf{0}, a] \cap [\mathbf{0}, b] \supseteq \{\mathbf{0}\}$, the family of chains in $[\mathbf{0}, a] \cap [\mathbf{0}, b]$ is nonempty. According to Zorn's lemma, there is a maximal chain C in $[\mathbf{0}, a] \cap [\mathbf{0}, b]$. According to the assumption, there is an upper bound $c \in [\mathbf{0}, a] \cap [\mathbf{0}, b]$ of C . Since the chain C is maximal, $c \in C$ is a maximal element of $[\mathbf{0}, a] \cap [\mathbf{0}, b]$.

(JPCU) \Rightarrow (L): See Theorem 3.7.

(L) \Rightarrow (CU): Let $a, b \in E$. The element $a \wedge b$ is an upper bound for every chain in $[\mathbf{0}, a] \cap [\mathbf{0}, b]$.

Let us present examples that show that the scheme of implications in the previous theorem cannot be improved in the sense that no other implications (except direct consequences of the transitivity) between given conditions hold.

4.2. Example. Let X be an infinite set, $y \notin X$, $E = \{\emptyset\} \cup \{\{x, y\} : x \in X\} \cup \{X \setminus \{x\} : x \in X\} \cup \{X \cup \{y\}\}$, $A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X \cup \{y\})$ is an infinite chain-finite concrete orthomodular lattice.

4.3. Example. Let X be an uncountable set, $E = \exp X$ with $A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X)$ is an orthocomplete concrete orthomodular lattice (it forms a Boolean algebra) such that there is an uncountable set of mutually orthogonal elements. Hence it is not chain-finite and every unital set of states on E is uncountable.

4.4. Example. Let X be a countable infinite set. Let E be a family of finite and cofinite subsets of X with the \oplus operation defined as the union of disjoint sets. Then $(E, \oplus, \emptyset, X)$ is a concrete orthomodular lattice (it forms a Boolean algebra) fulfilling the condition (JPCU) (every state on a Boolean algebra is Jauch–Piron, there is a countable unital set of states carried by points) that is not orthocomplete.

4.5. Example. Let X be a 6-element set. Let E be the family of even-element subsets of X with the \oplus operation defined as the union of disjoint sets from E . Then $(E, \oplus, \emptyset, X)$ is a finite concrete orthomodular poset that is not a lattice.

4.6. Example. Let X, Y be disjoint infinite countable sets,

$$E_0 = \{A \subseteq (X \cup Y) : \text{card}(A \cap X) = \text{card}(A \cap Y) \text{ is finite}\},$$

$$E = E_0 \cup \{(X \cup Y) \setminus A : A \in E_0\},$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X \cup Y)$ is a concrete orthomodular poset with the maximality property. Let $X = \{x_n : n \in \mathbb{N}\}$, $y_0 \in Y$, $f : X \rightarrow Y \setminus \{y_0\}$ be a bijection, $A = (X \cup Y) \setminus \{x_1, f(x_1)\}$, $B = (X \cup Y) \setminus \{x_1, y_0\}$. Then the chain $\{\{x_2, \dots, x_n, f(x_2), \dots, f(x_n)\} : n \in \mathbb{N} \setminus \{1\}\}$ in $[\emptyset, A] \cap [\emptyset, B]$ does not have an upper bound in $[\emptyset, A] \cap [\emptyset, B]$, hence the condition (CU) from Theorem 4.1 is not fulfilled.

Let us remark that not all effect algebras have the maximality property (see Example 3.8).

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