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Representations of Orthomodular Structures

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Abstract

There are various possibilities how to represent orthomodular structures. One attempt (topological) follows the famous Stone construction for representation of Boolean algebras and leads to a representation by means of clopen subsets of a closure space. This enables (as we will show) to visualize some algebraic constructions and proofs.

Another attempt (graphical) pays attention to orthogonality relations and leads to orthogonality diagrams, more compact Greechie diagrams and their generalizations and to dual diagrams. These representations are useful especially if we study probability measures (states). We will give several examples for the so-called Kochen–Specker type constructions, i.e., such orthomodular lattices realizable in a Hilbert space that do not have any two-valued measure.

1 Introduction and basic notions

There are several possibilities how to represent posets (partially ordered sets), orthoposets (orthocomplemented partially ordered sets) or orthomodular posets. Let us mention Hasse diagrams or representation by means of functions. We will give a short survey of two kinds of representations useful for orthomodular structures — structures used in quantum theories. Both these attempts are useful in the study of an axiomatics of quantum theories.

Let us start with basic definitions.

Definition 1.1 An *orthoposet* is a structure $(P, \le, ', 0, 1)$ such that (for every $a, b \in P$):

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(1) $\leq$ is a partial ordering, $0 \leq a \leq 1$;
(2) $'$ is an *orthocomplementation*, i.e.:
 (a) $a'' = a$;
 (b) $a \leq b$ implies $b' \leq a'$;
 (c) $a \vee a' = 1$.

Elements a, b of an orthoposets are called *orthogonal* (denoted by $a \perp b$) if $a \leq b'$. An orthoposet is called *orthomodular poset* if the following conditions are fulfilled (for every $a, b \in P$):
(3) $a \vee b$ exists for $a \perp b$;
(4) (*orthomodular law*) $b - a$ exists in P for $a \leq b$, where $b - a$ is defined by the properties: $(b - a) \perp a$ and $(b - a) \vee a = b$.

Basic examples of orthomodular posets are Boolean algebras and sets of closed subspaces of a Hilbert space. We will also speak about *orthomodular lattice* if the orthomodular poset is a lattice.

Another important notion is the notion of a measure.

Definition 1.2 A *measure* on an orthoposet P is a mapping $m: P \to [0, 1]$, such that (for every $a, b \in P$):
(1) $m(1) = 1$;
(2) $m(a \vee b) = m(a) + m(b)$ for $a \perp b$ whenever $a \vee b$ exists.

It is easy to see that $m(0) = 0$, $m(a) + m(a') = 1$ and that the orthomodular law implies monotonicity ($m(a) \leq m(b)$ for $a \leq b$).

We will have a special interest on two-valued measures (i.e., measures with values in $\{0, 1\}$). There might be a lot of them (e.g. in a Boolean algebra), while, on the other side, the set of closed subspaces of a Hilbert space with dimension greater than 2 admits no two-valued measure.

2 Set representations

Following the famous Stone construction of the set representation we can find a set representation for an arbitrary orthoposet. However, we obtain a closure space instead of a topological space and not all suprema are represented by unions. The following theorem is proved in [6].

Theorem 2.1 *For every orthoposet P and for every Boolean subalgebra B of P there is a set representation of P on a closure space X ($\leq\, = \,\subseteq$, $'\, = {}^c$, $0 = \emptyset$, $1 = X$) by clopen subsets (all if P is a lattice) of X such that the image of B corresponds to its Stone representation.*

Let us present a result from [7] which can be easily proved using this representation.

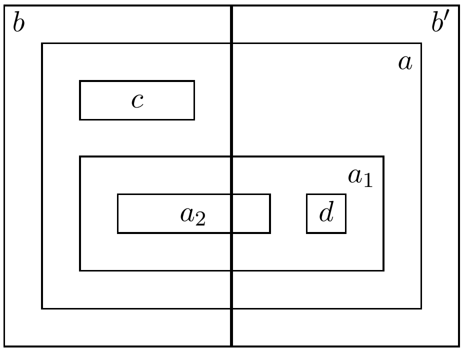

Figure 1: Illustration of the proof of Theorem 2.2.

Theorem 2.2 *Every orthomodular poset P fulfilling the conditions (for every $a, b \in P$):*

(1) $[0, a] \cap [0, b]$ *has a maximal element,*

(2) $a \wedge b = a \wedge b' = 0$ *implies* $a = 0$,

is a Boolean algebra.

Proof. (See Fig. 1.) Let us take a set representation of P on a set X. Let $a, b \in P$. We can represent them by sets, $b \cup b' = X$. According to condition (2), there is a maximal element c of $[0, a] \cap [0, b]$, i.e., $c \subseteq a \cap b$. Let us put $a_1 = a - c$. According to condition (2) again, there is a maximal element d of $[0, a_1] \cap [0, b']$, i.e., $d \subseteq a_1 \cap b'$. Let us put $a_2 = a_1 - d$. It is easy to see that $a_2 \wedge b = a_2 \wedge b' = 0$, hence, according to condition (1), $a_2 = 0$, $a_1 = d$ and $a = c \vee d$. The element a is cutted by b additively, elements a an b are compatible. Since we have started with an arbitrary pair of elements, P is a Boolean algebra.

Let us remark that the the second condition of Theorem 2.2 is a weak form of distributivity: $a \wedge b = a \wedge b' = 0$ implies $a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b')$. It should be also noted that the proof can be used for more general structures with "difference", e.g. for so-called difference posets or orthoalgebras.

The two conditions of Theorem 2.2 are generalizations of various conditions used in analogous results (see [7]).

Theorem 2.3 *An orthomodular poset P fulfills the condition (1) of Theorem 2.2 if one of the following conditions is fulfilled:*

(L) *P is a lattice.*

(OC) *P is orthocomplete (i.e., suprema of sets of mutually orthogonal elements of P exist in P).*

(JP) *Every measure on P is Jauch–Piron (i.e., if $m(a) = m(b) = 1$, then there is a $c \leq a, b$ such that $m(c) = 1$) and there is a countable* *unital set of measures (i.e., for every $a \neq 0$ there is a measure m from this set such that $m(a) = 1$).*

Theorem 2.4 *An orthomodular poset P fulfills the condition (2) of Theorem 2.2 if one of the following conditions is fulfilled (for every $a, b \in P$):*

(AB) $a \wedge b = 0$ *implies* $a \perp b$.

(f1) *There is a full set (if $a \not\leq b$ then there is an m from this set such that $m(a) \not\leq m(b)$) of two-valued Jauch–Piron measures.*

(f2) *There is a full set of subadditive measures (for a, b there is a $c \leq a, b$ such that $m(a) + m(b) \leq 1 + m(c)$).*

(f3) *There is a full set of measures with the property:* $a \wedge b = 0$ *implies* $m(a) + m(b) \leq 1$.

(u2) *There is a unital set of subadditive measures.*

(u3) *There is a unital set of measures with the property:* $a \wedge b = 0$ *implies* $m(a) + m(b) \leq 1$.

(u4) *There is a unital set of measures with the property:* $a \wedge b = 0, m(a) = 1$ *implies* $m(b) = 0$.

(u5) *There is a unital set of measures with the property:* $a \wedge b = 0, m(a) = 1$ *implies* $m(b) < \frac{1}{2}$.

(w2) *There is a weakly unital set (for every $a \neq 0$ there is an m from this set such that $m(a) > \frac{1}{2}$) of subadditive measures.*

3 Orthogonality diagrams

Another possibility is represent the orthomodular poset by some kind of an orthogonality diagram. These representations enables an easy study of measures. We will illustrate these representations on the so-called Kochen–Specker type constructions—examples of such finite orthomodular lattices of closed subspaces of a 3-dimensional Hilbert space that do not have any two-valued measure.

It was known that, according to the well-known Gleason theorem, the lattice of all closed subspaces of a Hilbert space with the dimension greater than 2 does not admit a two-valued measure. However, the Gleason theorem uses essentially an infinite number of elements. It was an interesting result to present a finite orthomodular lattice realizable in a 3-dimensional Hilbert space without any two-valued measure [2].

First we can represent atoms of an orthoposet by points and connect points corresponding to orthogonal elements by lines. At Fig. 2 there is an example [2] of a finite orthomodular poset realizable in a 3-dimensional Hilbert space without any two-valued state given by Kochen and Specker. (In fact it is not a one-to-one representation, because some different points at the diagram represent the same element of corresponding orthomodular poset, see [2, 5].)

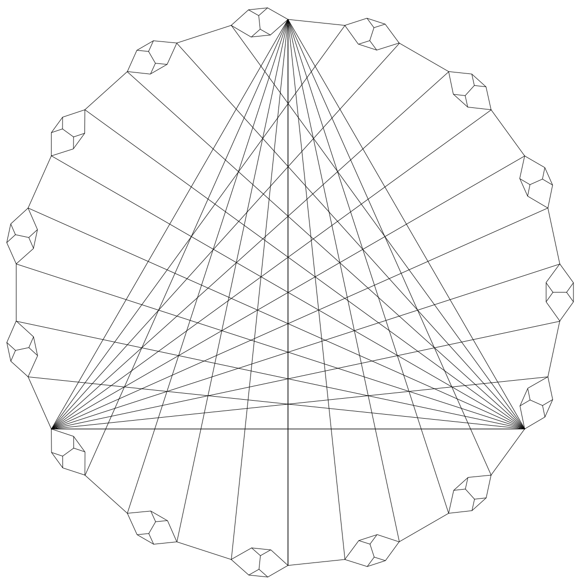

Figure 2: Orthogonality diagram of an orthoposet in a 3-dimensional Hilbert space without a two-valued state [2].

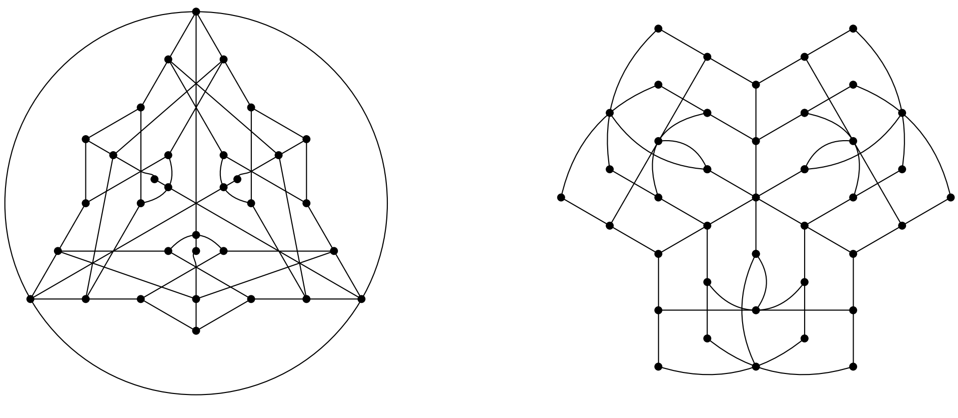

Figure 3: Greechie and dual diagrams [8] of an orthomodular poset in a 3-dimensional Hilbert space without a two-valued state given by Peres [4]. (Some unnecessary points are omitted.)

More compact representation connects by a smooth curve (usually by a line) all mutually orthogonal atoms. We obtain the so-called Greechie diagram [1]. This representation is also suitable to construct orthomodular structures: there are various conditions on the diagrams [1, 3] which ensure that the diagram is a diagram of an orthomodular structure. In Fig. 3 there is such a representation [8] of an orthomodular poset without a two-valued state realisable in a 3-dimensional Hilbert space given by Peres [4]. (Points corresponding to atoms are marked explicitly.)

Sometimes it is also useful to use dual diagrams: we can interchage the roles of points and curves. An example of such a representation [8] is given in Fig. 3—it is the representation of the same orthomodular poset as in the previous example.

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