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Partially additive measures and set representations of orthoposets

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Abstract

We introduce and study partially additive (i.e., additive on suitable substructures) measures on orthoposets. This generalizes several previous attempts to obtain extension theorems and a proper set representation. We use set representations for completions of orthoposets and for investigation of pointwise carrying of homomorphisms.

Introduction

Since the Stone representation of Boolean algebras (by means of clopen subsets of totally disconnected compact Hausdorff topological spaces) it has been natural to look for a topological representation of algebraic structures. Here we do this for orthoposets. Ideally, we would like to find a set representation of an orthoposet such that the least element corresponds to the empty set, the partial ordering corresponds to the inclusion relation, the orthocomplementation corresponds to the set theoretical complementation and (finite) orthogonal suprema correspond to set theoretical unions. However, it is known that such a representation exists only for orthoposets with a full set of two-valued measures (see [20] for orthomodular posets). Thus, it is necessary to give up the latter correspondence and look for a weaker one.

Previously, the investigation went in two directions. The first line of investigation was based on the concept of an M-base [11, 12, 9] and led to a set representation of an orthoposet. The second line was the effort to find a 'better' representation of orthomodular posets in the sense that some (finite) orthogonal suprema correspond to unions. As a result, the representation then corresponded to the Stone representation for some Boolean subalgebras of the orthomodular posets in question ([20, 8, 1] for the center, [19] for a given Boolean subalgebra).

In this paper we present a common generalization of all these results. The character of the questions investigated here led us to introducing and analysing so-called partially additive measures. Apart from results explicitly needed for the representation theorem, we also obtain extension theorems for those measures strengthening thus the results of [15, 1, 19]. As a consequence of the representation theorem we obtain stronger versions of results on the completion of orthoposets [10, 4, 16]. At the end, we consider the pointwise carrying of homomorphisms and formulate a few open questions.

1. Basic notions

Definition 1.1. An *orthoposet* is a triple $(P, \leq, ')$, such that

(1) $(P, \leq)$ is a partially ordered set with a least element, 0, and a greatest element, 1,

(2) $' : P \to P$ is an *orthocomplementation*, i.e., for any $a, b \in P$ we have

(a) $a'' = a$,

(b) $a \leq b \Rightarrow b' \leq a'$,

(c) $a \vee a' = 1$.

Let us call elements $a, b \in P$ *orthogonal* (denoted by $a \perp b$) if $a \leq b'$, and let us denote by $\mathrm{OS}(P)$ the set of all finite subsets of P of mutually orthogonal elements that have a supremum in P.

An orthoposet $(P, \leq, ')$ is called an *ω-orthocomplete poset* if $a \vee b$ exists for any pair $a, b \in P$ of orthogonal elements. An ω-orthocomplete poset $(P, \leq, ')$ is called an *orthomodular poset* if the *orthomodular law* is valid in P, i.e., $b = a \vee (b \wedge a')$ for every $a, b \in P$ such that $a \leq b$.

An orthoposet $(P, \leq, ')$ is called *Boolean* if $a \perp b$ for every $a, b \in P$ such that $a \wedge b = 0$.

An orthoposet $(P, \subset, ^c)$ with $P \subset \exp X$ for some set X is called a *set orthoposet* (denoted by (X, P)). A set orthoposet (X, P) is called *concrete* if $\bigvee R = \bigcup R$ for every $R \in \mathrm{OS}(P)$.

Let us note that Boolean orthoposets are concrete and Boolean ortholattices are exactly Boolean algebras (see e.g. [18]).

Dealing with an orthoposet $(P, \leq, ')$, we shall shortly denote it by P if there is no danger of misunderstanding.

Definition 1.2. Let $(P, \leq, ')$ be an orthoposet and let P_1 be a subset of P that contains elements $0, 1$ and a' for any $a \in P_1$. Then $(P_1, \leq|_{P_1}, '|_{P_1})$ is called a *suborthoposet* of $(P, \leq, ')$. A suborthoposet P_1 of an orthoposet P is called a *subortholattice* if it is a lattice and if the lattice operations on P_1 are restrictions of those in P.

Definition 1.3. Let $(P_1, \leq, ')$, $(P_2, \leq, ')$ be orthoposets. A mapping $h : P_1 \to P_2$ is called a *homomorphism* if for any $a, b \in P_1$ the following conditions hold:
(1) $h(0) = 0$,
(2) $h(a') = h(a)'$,
(3) $a \leq b \Rightarrow h(a) \leq h(b)$.
A homomorphism h is called an *embedding* if
(4) $h(a) \leq h(b) \Rightarrow a \leq b$ for any $a, b \in P_1$.
An embedding h is called an *isomorphism* if $h(P_1) = P_2$.
A homomorphism h is called an *orthohomomorphism* if
(5) $h(\bigvee R) = \bigvee h(R)$ for any $R \in \mathrm{OS}(P_1)$.
An orthohomomorphism h is called an *orthoembedding* if it is an embedding and if
(6) $\bigvee R \in h(P_1)$ for any $R \in \mathrm{OS}(P_2)$ such that $R \subset h(P_1)$.

Observe that 'orthoisomorphism' would not mean anything else than isomorphism.

2. Partially additive measures

In the Stone representation we can equivalently use various objects: ultrafilters, prime ideals, homomorphisms into the two-element Boolean algebra or two-valued measures. In some applications of the theory of orthostructures (e.g. in the quantum logic theory) the natural object is a two-valued measure (state) or its generalization. We will introduce it in the next definition. It should be noticed that the definition covers various approaches (see [11, 20, 14, 19]).

Definition 2.1. Let $(P, \leq, ')$ be an orthoposet and let $\mathcal{R} \subset \mathrm{OS}(P)$. By a *partially additive measure on P with respect to $\mathcal{R}$* (abbr. *$\mathcal{R}$-measure*) we mean a mapping $m : P \to [0, 1]$ such that
(1a) $m(1) = 1$,
(1b) $\forall a \in P : \ m(a) + m(a') = 1$,
(1c) $\forall R \in \mathcal{R} : \ \sum_{a \in R} m(a) = m(\bigvee R)$,
(2) $\forall a, b \in P : a \leq b \Rightarrow m(a) \leq m(b)$.
A partially additive measure (i.e., with respect to $\emptyset$) m is called *Jauch-Piron* if for any $a, b \in P$ with $m(a) = m(b) = 1$, there is a $c \in P$ such that $c \leq a, b$ and $m(c) = 1$.
A set M of some partially additive measures is called *full* if for any $a, b \in P$ with $a \not\leq b$ there is an $m \in M$ such that $m(a) \not\leq m(b)$.

Obviously, the larger the set $\mathcal{R}$ is, the more involved the $\mathcal{R}$-measure is. An $\mathrm{OS}(P)$-measure on an orthoposet P is simply called a *measure*.

If the orthoposet P is ω-orthocomplete, we may restrict ourselves in the condition (1c) to the two-element subsets of P. If the orthoposet P is orthomodular and $\mathcal{R} = \mathrm{OS}(P)$, the condition (2) follows from the condition (1c) and from the orthomodular law.

It is an easy observation that every two-valued Jauch-Piron partially additive measure is a measure.

Before stating several examples dealt with in the literature, let us recall that by the *center* of an orthomodular poset P (denoted by $C(P)$) we mean the set of all elements $c \in P$ such that for any $a \in P$ there are mutually orthogonal elements $a_1, c_1, b \in P$ such that $c = c_1 \vee b$ and $a = a_1 \vee b$. The center of an orthomodular poset is easily shown to be Boolean subalgebra (see e.g. [6]).

Examples 2.2. (1) $\mathcal{R} = \emptyset$. Each $\mathcal{R}$-measure m corresponds to the M-base $m^{-1}(1)$. See [11, 12, 9].

(2) $\mathcal{R} = \mathrm{OS}(C(P))$. See [20, 8] for an orthomodular poset P.

(3) $\mathcal{R} \subset \mathrm{OS}(P)$ such that every $R \in \mathcal{R}$ contains at most one non-central element. See [14, 1] for an orthomodular poset P.

(4) $\mathcal{R} = \mathrm{OS}(B)$, where B is a Boolean subalgebra of P. See [19] for an orthomodular poset P.

(5) $\mathcal{R} = \mathrm{OS}(P)$. See [20, p. 262], for an orthomodular poset P.

3. Extensions of measures

In [14], Pták characterized orthomodular posets such that every measure on each of its Boolean subalgebras can be extended to the entire orthomodular poset. Using the same technique, in [14, 1, 19] it is shown that any (two-valued) measure on a Boolean subalgebra of an orthomodular poset can be extended to a suitable (two-valued) partially additive measure on a given orthomodular poset (see Examples 2.2(3) and 2.2(4)). Here we present generalizations of these results. We shall need the following notion (for any $a, b \in P$ we shall use the standard notation $[a, b] = \{c \in P;\ a \leq c \leq b\}$).

Definition 3.1. Let P be an orthoposet and let $\mathcal{R} \subset \mathrm{OS}(P)$. The set $I \subset P$ is called a *partial ideal with respect to* $\mathcal{R}$ (abbr. $\mathcal{R}$*-ideal*) if the following holds:

(1) $[0, a] \subset I$ for every $a \in I$,

(2) $\bigvee R \in I$ for every $R \in \mathcal{R}$ with $R \subset I$.

An $\mathcal{R}$-ideal is called a *proper* $\mathcal{R}$*-ideal* if it does not contain any pair of orthocomplemented elements.

A proper $\mathcal{R}$-ideal is called a *maximal* $\mathcal{R}$*-ideal* if there is no other proper $\mathcal{R}$-ideal containing it.

A proper $\mathcal{R}$-ideal is called a *prime* $\mathcal{R}$*-ideal* if it contains exactly one element out of every pair of orthocomplemented elements.

Lemma 3.2. *Suppose that P is an orthoposet. Then there is a one-to-one correspondence between two-valued $\mathcal{R}$-measures and prime $\mathcal{R}$-ideals given by the mapping $m \mapsto m^{-1}(0)$.*

Proof. Obvious. $\square$

It is easy to see that every $\mathcal{R}$-ideal is contained in a maximal $\mathcal{R}$-ideal (Zorn's lemma) and that every prime $\mathcal{R}$-ideal is a maximal $\mathcal{R}$-ideal. The validity of the converse inclusion will be in our interest here. Let us first introduce a notion that is a generalization of two important properties.

Definition 3.3. Let P be an orthoposet. We say that a family $\mathcal{R} \subset \mathrm{OS}(P)$ has the *extension property* if the following holds:

For every proper $\mathcal{R}$-ideal I on P and for every $a \in P$ with $I \cap \{a, a'\} = \emptyset$ there is a proper $\mathcal{R}$-ideal J on P such that $I \cup \{a\} \subset J$.

Lemma 3.4. *Suppose that P is an orthoposet and that $\mathcal{R} \subset \mathrm{OS}(P)$ has the extension property. Then*

(1) *Every maximal $\mathcal{R}$-ideal on P is a prime $\mathcal{R}$-ideal on P.*

(2) *The set of all two-valued $\mathcal{R}$-measures on P is full.*

Proof. (1) Obvious.

(2) Suppose that $a, b \in P$ and $a \not\leq b$. Then $[0, b]$ is a proper $\mathcal{R}$-ideal and, according to the extension property, $[0, b] \cup \{a'\}$ is contained in a proper $\mathcal{R}$-ideal, J. Then J is contained in a maximal $\mathcal{R}$-ideal I that is (part (1)) a prime $\mathcal{R}$-ideal. According to Lemma 3.2, the prime $\mathcal{R}$-ideal I corresponds to a two-valued $\mathcal{R}$-measure m such that $m(a) = 1$ and $m(b) = 0$. $\square$

Before stating our general extension theorem, let us recall properties of families $\mathcal{R}$ from Examples 2.2.

Proposition 3.5. *Suppose that P is an orthoposet.*

(1) $\emptyset$ *has the extension property.*

(2) $\mathrm{OS}(C(P))$ *has the extension property, provided P is orthomodular.*

(3) $\mathcal{R} \subset \mathrm{OS}(P)$, *where every $R \in \mathcal{R}$ contains at most one non-central element, has the extension property, provided P is orthomodular.*

(4) $\mathrm{OS}(B)$ *has the extension property for any Boolean subalgebra B of P.*

Proof. (1) Obvious.

(2) See [20, Proposition 1.2].

(3) See [14, Lemma 1] (the orthomodularity of P seems to be essential).

(4) See [19, Proposition 2.4] (there was no need for orthomodularity or ω-orthocompleteness of P). $\square$

Since there are orthomodular lattices without a measure (see [5]) there is no chance of an analogous result for $\mathcal{R} = \mathrm{OS}(P)$ (see Lemma 3.4).

Theorem 3.6. *Suppose that P is an orthoposet, L is a subortholattice of P, $\mathcal{R} \subset \mathrm{OS}(P)$ such that each maximal $\mathcal{R}$-ideal is a prime $\mathcal{R}$-ideal. Then for any two-valued Jauch-Piron measure m on L there is a two-valued $\mathcal{R}$-measure $\widetilde{m}$ on P such that $\widetilde{m}|L = m$.*

Proof. The set $J = m^{-1}(0)$ is a (lattice) prime ideal on L, the set $I_0 = \bigcup\{[0,a];\ a \in J\}$ is an $\mathrm{OS}(P)$-ideal on P, hence it is an $\mathcal{R}$-ideal. According to the assumptions, there is a prime $\mathcal{R}$-ideal $I \supset I_0$. Since $I \cap L = J$, the corresponding two-valued measure $\widetilde{m}$ satisfies $\widetilde{m}|L = m$. □

Let us note that on a Boolean algebra every measure is Jauch-Piron.

Theorem 3.7. *Suppose that P is an orthoposet, B is a Boolean subalgebra of P, $\mathcal{R} \subset \mathrm{OS}(P)$ such that each maximal $\mathcal{R}$-ideal is a prime $\mathcal{R}$-ideal. Then for any measure m on B there is an $\mathcal{R}$-measure $\widetilde{m}$ on P such that $\widetilde{m}|B = m$.*

Proof. The set of all $\mathcal{R}$-measures on P (denoted by $S_{\mathcal{R}}(P)$) is a closed subset of a topological space $[0,1]^P$ (with a product topology), hence compact (Tichonov's theorem).

Let us denote by $\mathcal{D}$ the set of all partitions of unity in P (i.e., the set of all finite subsets D of non-zero mutually orthogonal elements such that $\bigvee D = 1$). Put $F_D = \{\bar{m} \in S_{\mathcal{R}}(P);\ \bar{m}|B \text{ is a measure on } B \text{ and } \bar{m}|D = m|D\}$ for every $D \in \mathcal{D}$. We shall show that $\mathcal{F} = \{F_D;\ D \in \mathcal{D}\}$ is a filter base consisting of nonempty closed subsets of $S_{\mathcal{R}}(P)$.

First, every set F_D is closed ('pointwise convergence'). Let D, E be two partitions of unity in B. Then $F_D \cap F_E \supset F_{(D\wedge E)\setminus\{0\}}$, where $(D \wedge E)\setminus\{0\} = \{d \wedge e;\ d \in D \text{ and } e \in E\}\setminus\{0\}$ is a partition of unity in B. Finally, let D be a partition of unity in B. For every $d \in D$ there is a two-valued measure m_d on B such that $m_d(d) = 1$ (it is well-known that the set of all two-valued measures on a Boolean algebra is full, it follows e.g. from Proposition 3.5(4) and from Lemma 3.4). According to Theorem 3.6, there is a two-valued $\mathcal{R}$-measure $\widetilde{m}_d$ on P such that $\widetilde{m}_d(d) = 1$. Hence $\sum_{d\in D} m(d)\widetilde{m}_d \in F_D$.

Thus, $\mathcal{F}$ is the base of a proper filter in a compact space and we have an $\mathcal{R}$-measure $\widetilde{m} \in \bigcap \mathcal{F}$. It follows immediately from the definition of $\mathcal{F}$ that $\widetilde{m}$ extends m. The proof is complete. □

It is easy to see from the above proof that the necessary and sufficient condition for the latter extension property is that the set of all such $\mathcal{R}$-measures on P whose restrictions to B are measures on B is *unital on B* (i.e., for every $a \in B\setminus\{0\}$ there is an $m \in S_{\mathcal{R}}(P)$ such that $m|B$ is a measure on B and $m(a) = 1$). In fact, we have proved that with given assumptions this condition is satisfied by the set of all two-valued $\mathcal{R}$-measures on P.

This observation gives immediately the following result (see also [15] for orthomodular posets).

Corollary 3.8. *Suppose that P is an orthoposet. Then the following statements are equivalent:*

(1) *The set of all measures on P is unital on P.*

(2) *For every Boolean subalgebra B of P and for every measure m on B there is a measure $\widetilde{m}$ on P such that $\widetilde{m}|B = m$.*

The following corollary we obtain for family $\mathcal{R}$ from Proposition 3.5(4).

Corollary 3.9. *Suppose that P is an orthoposet, B, B_1 are Boolean subalgebras of P and that m_1 is a measure on B_1. Then there is a $\mathrm{OS}(B)$-measure m on P such that $m|B_1 = m_1$.*

If P is a Boolean algebra and if we take $B = P$, the above corollary gives a topological proof of a well-known result of Horn and Tarski [7].

4. Set representations of orthoposets

We shall give a representation theorem that summarises and generalizes results from [8, 1, 19, 20].

In the Stone representation we represent a Boolean algebra by clopen sets in a topological space. In the present context it is useful to consider a more general underlying space (see Theorem 4.2, parts (4) and (10)).

Definition 4.1. A *closure space* is a pair $(M,^-)$ such that $M \neq \emptyset$ and $^- : \exp M \to \exp M$ is a *closure operation*, i.e.,

(1) $\overline{\emptyset} = \emptyset$,

(2) $A \subset \overline{A}$,

(3) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$,

(4) $\overline{\overline{A}} = \overline{A}$.

A set $A \subset M$ is called *closed* if $\overline{A} = A$, *open* if $M \setminus A$ is closed, *clopen* (denoted by $A \in \mathrm{CO}(M)$) if both A and $M \setminus A$ are open.

A family $\mathcal{B} \subset \exp M$ of open sets is called a *base of open sets* if for any open $A \subset M$ there is a $\mathcal{B}_1 \subset \mathcal{B}$ such that $A = \bigcup \mathcal{B}_1$.

A closure space $(M,^-)$ is called *Hausdorff* if any pair of points from M is separated by disjoint open sets, *compact* if any open covering of M has a finite subcovering, *0-dimensional* if $\mathrm{CO}(M)$ is a base of open sets.

The union of two closed sets (the intersection of two open sets, resp.) in closure space need not be closed (open). On the other hand, the intersection of any family of closed sets (the union of any family of open sets, resp.) has to be closed (open).

If we replace the condition (3) by a stronger condition

(3') $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

we obtain the definition of a *topological space*.

Every 0-dimensional Hausdorff closure space is *totally disconnected*, i.e., any pair of points in it is separated by disjoint clopen sets.

Every family $\mathcal{B} \subset \exp M$ such that $\bigcup \mathcal{B} = M$ is a base of open sets for some closure space $(M, ^-)$ (we put $\overline{A} = M \setminus \bigcup \{B \in \mathcal{B};\ B \cap A = \emptyset\}$ for any $A \subset M$) and a subbase for the *associated* topological space. According to Alexander's sub-base theorem, the closure space is compact if and only if the associated topological space is compact.

Theorem 4.2. *Suppose that $(P, \leq, ')$ is an orthoposet, $\mathcal{R} \subset \mathrm{OS}(P)$, M is a nonempty set of two-valued $\mathcal{R}$-measures on P, $h : (P, \leq, ') \to (\exp M, \subset, ^c)$, where $h(a) = \{m \in M;\ m(a) = 1\}$ for any $a \in P$ and $h(P)$ is the base of open sets in $(M, ^-)$. Then:*

(1) *h is a homomorphism.*

(2) *$h(P) \subset \mathrm{CO}(M)$.*

(3) *$(M, ^-)$ is a 0-dimensional Hausdorff closure space.*

(4) *If $\mathcal{A} \subset h(P)$ and $\bigvee \mathcal{A}$ exists in $(h(P), \subset)$, then $\bigvee \mathcal{A} = \overline{\bigcup \mathcal{A}}$.*

(5) *$h(\bigvee R) = \bigvee h(R) = \bigcup h(R)$ for every $R \in \mathcal{R}$.*

(6) *h is an embedding iff M is full.*

(7) *h is an orthohomomorphism iff each $\mathcal{R}$-measure in M is a measure.*

(8) *h is an orthoembedding iff P is an orthomodular poset and M is a full set of two-valued measures on P.*

(9) *If M is the set of all two-valued $\mathcal{R}$-measures, then $(M, ^-)$ is a compact closure space.*

(10) *If P is an ortholattice, M is the set of all two-valued $\mathcal{R}$-measures and is full, then $h(P) = \mathrm{CO}(M)$.*

(11) *If each $m \in M$ is Jauch-Piron then $(M, ^-)$ is a topological space. On the other hand, if $(M, ^-)$ is a topological space and M is full, then each $m \in M$ is Jauch-Piron.*

Proof. (1) Obvious.

(2) For any $a \in P$ the set $h(a)$ is open in $(M, ^-)$. Since $h(a) = h(a')^c$, it is closed, too.

(3) Since $h(P) \subset \mathrm{CO}(M)$ is a base of open sets, the closure space $(M, ^-)$ is 0-dimensional. Suppose that $s_1, s_2 \in M$, $s_1 \neq s_2$. Then there is an $a \in P$ such that $s_1(a) \neq s_2(a)$. Hence $h(a), h(a')$ are disjoint clopen sets that separate s_1, s_2.

(4) According to the definition of the closure space $(M, ^-)$, we have $\bigvee \mathcal{A} = \bigcap \{B \in h(P);\ B \supset \bigcup \mathcal{A}\} = \bigcap \{B \subset M;\ B \text{ closed and } B \supset \bigcup \mathcal{A}\} = \overline{\bigcup \mathcal{A}}$.

(5) Since h is a homomorphism, $h(\bigvee R) \supset \bigcup h(R)$. Suppose that $m \in h(\bigvee R)$. Then $1 = m(\bigvee R) = \sum_{a \in R} m(a)$, hence $m \in \bigcup h(R)$.

(6) Obvious.

(7) Suppose that each $\mathcal{R}$-measure in M is a measure. According to (1) and (5), h is an orthohomomorphism.

Suppose now that h is an orthohomomorphism and that $m \in M$. For any $R \in \mathrm{OS}(P)$ with $m(\bigvee R) = 1$ we have $m \in h(\bigvee R) = \bigcup h(R)$. Since $h(R)$ consists of mutually disjoint sets, $m \in h(a)$ for exactly one $a \in R$, hence $1 = \sum_{a \in R} m(a)$. The $\mathcal{R}$-measure m is a measure.

(8) Suppose that P is ω-orthocomplete poset and that M is a full set of two-valued measures on P. According to (6) and (7), h is orthohomomorphism and embedding.

Suppose now that h is an orthoembedding. According to (6) and (7), M is a full set of two-valued measures. Since $\exp M$ is an ω-orthocomplete poset, P is ω-orthocomplete, too. Every ω-orthocomplete poset with a full set of two-valued measures is orthomodular—indeed, for every $a, b \in P$ with $a \leq b$ we obtain $m(a \vee (b \wedge a')) = m(a \vee (b' \vee a)') = m(a) + 1 - (1 - m(b) + m(a)) = m(b)$.

(9) The closure space $(M, ^-)$ is compact iff the associated topological space is compact. The associated topological space is a closed subspace of $\{0,1\}^P$ (with the product topology) that is compact (Tichonov's theorem).

(10) Suppose that $A \in \mathrm{CO}(M)$. Since A is open, there is a set $F \subset P$ such that $A = \bigcup h(F)$. Since A is closed, hence compact (part (9)), we can choose a finite covering $h(F_0)$. According to (6) and (4), $h(\bigvee F_0) = \bigvee h(F_0) = \overline{\bigcup h(F_0)} = \overline{A} = A$ ($\bigvee h(F_0)$ taken in $(h(P), \subset)$).

(11) Suppose that each $m \in M$ is Jauch-Piron. Then for any $a, b \in P$ and for any $m \in h(a) \cap h(b)$ there is a $c \in P$ such that $m \in h(c) \subset h(a) \cap h(b)$. Hence $(M, ^-)$ is a topological space.

Suppose now that $(M, ^-)$ is a topological space and that M is full. Then for any $m \in M$ and for any $a, b \in P$ with $m(a) = m(b) = 1$ we have $m \in h(a) \cap h(b)$. Since $h(a) \cap h(b)$ is open, there is a $c \in P$ such that $m \in h(c) \subset h(a) \cap h(b)$. It means that $m(c) = 1$ and, according to (6), $c \leq a, b$. $\square$

The following consequence of the representation theorem and Proposition 3.5 is in our general context one of the main results of this paper.

Corollary 4.3. *Suppose that $(P, \leq, ')$ is an orthoposet, B is a Boolean subalgebra of P, $\mathcal{R} = \mathrm{OS}(B)$. Then there is a set representation $(h(P), \subset, ^c)$ of $(P, \leq, ')$ by means of clopen sets in a 0-dimensional compact Hausdorff closure space such that the image of B is 'almost' its Stone representation (i.e., the (finite) suprema in B correspond to set theoretical unions). Moreover, if P is an ortholattice, then we can ensure that the representation contains all clopen sets.*

We have shown that every orthoposet has a set representation, i.e., it can be embedded into a Boolean algebra. The following corollary generalizes a result of [20].

Corollary 4.4. (1) *An orthoposet P can be embedded into a Boolean algebra by an orthohomomorphism (i.e., it has a concrete representation) iff the set of all two-valued measures on P is full.*

(2) *An orthoposet P can be orthoembedded into a Boolean algebra iff P is an orthomodular poset with a full set of two-valued measures.*

Proof. Suppose that P is an orthoposet and that h is an embedding of P to a Boolean algebra. Every Boolean algebra can be considered as a subalgebra of $\exp X$ for some set X. Every $x \in X$ may be identified with a two-valued partially additive measure m_x on P defined in a such a way that for any $a \in P$ we have $m_x(a) = 1$ iff $x \in h(a)$. Hence, the embedding h may be considered of the form from Theorem 4.2. The rest follows from Theorem 4.2, parts (7) and (8). $\square$

5. Completions of orthoposets

In this section we shall show several consequences of Theorem 4.2(4). First of all, let us state basic definitions.

Definition 5.1. An ortholattice L is called *complete* if $\bigvee A$ exists in L for every $A \subset L$.

Let P be an orthoposet and let L be a complete ortholattice. An embedding $h : P \to L$ is called *completion* if $h(\bigvee A) = \bigvee h(A)$ for every $A \subset P$ such that $\bigvee A$ exists in P.

Standard methods for a completion of orthoposets are the completion by cuts and the completion by using the orthogonality relation [2, 10, 4]. Using these methods we (can) obtain a set representation such that suprema correspond to set-theoretic unions, whereas the orthocomplementation is more complicated. Here we obtain a completion such that the orthocomplementation corresponds to the set-theoretic complementation and suprema correspond to closures of set-theoretic unions (if we use an appropriate closure operation). Moreover, we shall generalize the result of Sekanina [16], who shows that every complete ortholattice is of the form of the set of all regularly open subsets of a closure space. Before stating our results, let us recall basics about regularly open sets and formulate a corollary of Theorem 4.2.

Definition 5.2. Let $(M, ^-)$ be a closure space. A set $A \subset M$ is called *regularly open* (denoted by $A \in \mathrm{RO}(M)$), if $A = \overline{(\overline{A}^c)}^c$.

It is known [16] that the set of all regularly open subsets of a closure space with the ordering given by inclusion and the orthocomplementation given by $A' = \overline{A}^c$ is a complete ortholattice. For every open set A the set $\overline{(\overline{A}^c)}^c$ is the smallest regularly open superset of A and is called the *regularisation of* A.

Every clopen set is regularly open, hence $\mathrm{CO}(M)$ is a suborthoposet of $\mathrm{RO}(M)$. It is known, too, that $\mathrm{CO}(M)$ and $\mathrm{RO}(M)$ are Boolean algebras in a topological space $(M, ^-)$.

Let us recall that a closure space is called *extremally disconnected* if $\mathrm{CO}(M) = \mathrm{RO}(M)$ (i.e., $\overline{A}$ is open for every open $A \subset M$). Every extremally disconnected Hausdorff closure space is totally disconnected.

Proposition 5.3. *Suppose that P, $\mathcal{R}$, M and h are as in Theorem 4.2 and that M is full. Then the embedding $h : P \to \mathrm{RO}(M)$ is a completion.*

Proof. Suppose that $A \subset P$ such that $\bigvee A$ exists in P. According to Theorem 4.2(6), $\bigvee h(A) = h(\bigvee A)$ in $h(P)$. According to Theorem 4.2(4), $\bigvee h(A) = \overline{\bigcup h(A)}$. Thus, making use of Theorem 4.2(2), $h(\bigvee A) = \overline{\bigcup h(A)} \in \mathrm{CO}(M) \subset \mathrm{RO}(M)$. The least regularly open set containing $\bigcup h(A)$ is the regularisation of $\bigcup h(A)$, i.e., $\bigvee h(A) = \left(\overline{\overline{\bigcup h(A)}^{c}}\right)^{c} = \overline{\bigcup h(A)} = h(\bigvee A)$. $\square$

Let us note that $\mathrm{RO}(S)$ in the above proposition is so-called *MacNeille completion* of P (see e.g. [3]; for every element $A \in \mathrm{RO}(S)$ there are $\mathcal{A}_1, \mathcal{A}_2 \subset h(P)$ such that $A = \bigvee \mathcal{A}_1 = \bigwedge \mathcal{A}_2$)

Corollary 5.4. *Every complete ortholattice is isomorphic to a family of all clopen subsets of a 0-dimensional extremally disconnected compact Hausdorff closure space.*

Proof. According to Theorem 4.2, Proposition 3.5 and Lemma 3.4, every ortholattice is isomorphic to $\mathrm{CO}(M)$ for some 0-dimensional compact Hausdorff closure space. Suppose that A is a regularly open set. Since $\mathrm{CO}(M)$ is a base of open sets, there is an $\mathcal{A} \subset \mathrm{CO}(M)$ such that $A = \bigcup \mathcal{A}$. Then $A = \bigvee \mathcal{A}$ in $\mathrm{RO}(M)$ and, according to Proposition 5.3, A is clopen. Hence $\mathrm{RO}(M) = \mathrm{CO}(M)$ and the proof is complete. $\square$

This result generalizes the well-known representation theorem for complete Boolean algebras. In comparison with [16] we have proved several properties of underlying closure space and ensured that the orthocomplementation in a set representation is set-theoretic.

The first part of the following corollary is well-known (see [10], here it is proved in a different way). The second part seems to be new and interesting in quantum logic theory. Let us note that, according to Theorem 4.2(10), every orthoposet with a full set of two-valued Jauch-Piron measures can be completed to a Boolean algebra. Since such an orthoposet has to be Boolean, the following statement is more general and shows that relatively large class of orthoposets can be completed to a Boolean algebra (see [13]).

Corollary 5.5. (1) *Every orthoposet can be completed to an ortholattice.*

(2) *The MacNeille completion of a Boolean orthoposet is a Boolean algebra.*

Proof. (1) It follows immediately from Propositions 5.3 and 3.5 and Lemma 3.4.

(2) Let us take the completion of a Boolean orthoposet P to an ortholattice $\mathrm{RO}(M)$ as in part (1). It suffices to prove that $\mathrm{RO}(M)$ is a Boolean ortholattice. Suppose that $A, B \in \mathrm{RO}(M)$ such that $A \cap B \neq \emptyset$. Since $h(P)$ is a base of open sets, there are $C, D \in h(P)$ such that $C \subset A$, $D \subset B$ and $C \cap D \neq \emptyset$. Since $h(P)$ is a Boolean orthoposet, there is an $E \in h(P) \setminus \{\emptyset\} \subset \mathrm{RO}(M)$ such that $E \subset C \cap D \subset A \cap B$. The proof is complete. $\square$

6. Homomorphisms of orthoposets

In this section we shall study the question when a homomorphism of orthoposets is carried by a point mapping. We shall need the following definitions.

Definition 6.1. Let (X_1, P_1) and (X_2, P_2) be set orthoposets.

We say that a homomorphism $h : P_1 \to P_2$ is *carried by a point mapping* $f : X_2 \to X_1$ if $h(A) = f^{-1}(A)$ for every $A \in P_1$.

We say that a two-valued partially additive measure m on a set orthoposet (X_1, P_1) is *carried by a point* $x \in X_1$ if $m(A) = 1$ iff $x \in A$ $(A \in P_1)$. The partially additive measure carried by a point x is denoted by m_x.

Definition 6.2. We say that a set $P \subset \exp X$ is *separating* on X if for each pair of points $x, y \in X$ there is an $A \in P$ such that $x \in A$ and $y \notin A$.

Every point $x \in X$ in a set orthoposet (X, P) carries a two-valued partially additive measure on P. A set orthoposet (X, P) is separating on X iff there is no pair of points of X that carry the same two-valued partially additive measure on P. If (X, P) is non-separating we may identify the points of X that carry the same two-valued partially additive measure and, as a result, we obtain a separating representation $(\widetilde{X}, \widetilde{P})$ of (X, P) $(\widetilde{X} \subset X, \widetilde{P} = \{A \cap \widetilde{X};\ A \in P\})$.

Every set orthoposet can be identified with its set representation (Theorem 4.2) by means of the set of all two-valued partially additive measures carried by a point. Let us call a *maximal set representation* (abbr. MSR) of an orthoposet such a separating set representation that each two-valued partially additive measure is carried by a point (i.e., the set representation by means of all two-valued partially additive measures).

Lemma 6.3. *Suppose that (X_1, P_1) and (X_2, P_2) are set orthoposets and that $h : P_1 \to P_2$ is a homomorphism carried by a point mapping $f : X_2 \to X_1$. Then $m_y \circ h = m_{f(y)}$ for all $y \in X_2$.*

Proof. For all $y \in X_2$ and all $A \in P_1$ the following statements are equivalent: $(m_y \circ h)(A) = 1,\ y \in h(A),\ f(y) \in A,\ m_{f(y)}(A) = 1.$ $\square$

Proposition 6.4. *Suppose that (X_1, P_1) and (X_2, P_2) are set orthoposets and that $h : P_1 \to P_2$ is a homomorphism. Let us denote by $\mathcal{D}(f)$ the set of all $y \in X_2$ for which the set $\{x \in X_1;\ m_x = m_y \circ h\}$ is nonempty. For every $y \in \mathcal{D}(f)$, let us choose some $f(y)$ from this set. Then the mapping $f : \mathcal{D}(f) \to X_1$ satisfies the following conditions:*

(1) *$f^{-1}(A) = h(A) \cap \mathcal{D}(f)$ for all $A \in P_1$; particularly, if $\mathcal{D}(f) = X_2$ then h is carried by the mapping f,*

(2) *if $h(P_1)$ is separating on X_2 then f is one-to-one,*

(3) *if $\mathcal{D}(f) = X_2$ and h is an isomorphism then $A = \bigwedge\{B \in P_1;\ B \supset f(h(A))\}$ for all $A \in P_1$.*

Proof. (1) We have $y \in f^{-1}(A)$ iff $y \in \mathcal{D}(f)$ and $f(y) \in A$. The latter condition is equivalent to each of the following conditions: $m_{f(y)}(A) = 1,\ (m_y \circ h)(A) = 1,\ y \in h(A)$.

(2) Suppose that $y, z \in X_2$ and that $f(y) = f(z)$. Then $m_y \circ h = m_z \circ h$, i.e., $s_y(h(A)) = s_z(h(A))$ for all $A \in P_1$. Since $h(P_1)$ is separating on X_2, we obtain $y = z$.

(3) According to part (1), $h(A) = f^{-1}(A)$, hence $f(h(A)) = A$. Suppose that $B \supset f(h(A))$. According to part (1), $h(B) = f^{-1}(B) \supset h(A)$, hence $B \supset A$. $\square$

Corollary 6.5. *(cf. [17, Section 11B]) Let (X_1, P_1) and (X_2, P_2) be set orthoposets. A homomorphism $h : P_1 \to P_2$ is carried by a point mapping iff for each two-valued partially additive measure m_y on P_2 (carried by a point $y \in X_2$) the two-valued partially additive measure $m_y \circ h$ on P_1 is carried by a point.*

Proof. It follows from Lemma 6.3 and Proposition 6.4(1). $\square$

Corollary 6.6. (1) *Every homomorphism of an MSR into a set orthoposet is carried by a point mapping.*

(2) *If there is an isomorphism of a separating set orthoposet (X, P) to its MSR $(\widetilde{X}, \widetilde{P})$ that is carried by a point mapping f, then f is a one-to-one mapping of $\widetilde{X}$ onto X.*

Proof. (1) It follows from Proposition 6.4(1).

(2) According to Proposition 6.4(2), the mapping f is one-to-one. Suppose that $x \in X$. Then $m_x \circ h^{-1}$ is a two-valued partially additive measure on $\widetilde{P}$ and hence it is carried by a point $y \in \widetilde{X}$. According to Lemma 6.3, $m_{f(y)} = m_y \circ h = m_x \circ h^{-1} \circ h = m_x$. Since P is separating, we obtain $x = f(y)$. Thus, $f(\widetilde{P}) = P$. $\square$

Corollary 6.7. *Suppose that (X, P) is a separating set orthoposet, $(\widetilde{X}, \widetilde{P})$ is its MSR and that $h : \widetilde{P} \to P$ is an isomorphism. Then there is a one-to-one mapping $f : X \to \widetilde{X}$ such that*

$h(A) = f^{-1}(A)$ *for all* $A \in \widetilde{P}$,
$h^{-1}(C) = \bigwedge\{B \in \widetilde{P};\ B \supset f(C)\}$ *for all* $C \in P$.

Proof. It follows from Proposition 6.4. $\square$

Let us recall that a mapping $f : (X_1, ^-) \to (X_2, ^-)$ ($(X_1, ^-)$ and $(X_2, ^-)$ are closure spaces) is called *continuous* if for each open set $A \subset X_2$ the set $f^{-1}(A)$ is open. It is called a *homeomorphism* if there exists f^{-1} and both f and f^{-1} are continuous.

It is easy to see that a continuous mapping on a closure space is also continuous on the associated topological space. Indeed, let (X_1, τ_1) and (X_2, τ_2) be topological spaces associated with $(X_1, ^-)$, $(X_2, ^-)$, $\mathcal{B}_2$ be a base of open sets for $(X_2, ^-)$, $f : (X_1, ^-) \to (X_2, ^-)$ be a continuous mapping; then $\mathcal{B}_2$ is a subbase of open sets for (X_2, τ_2) and $f^{-1}(A)$ is open for every $A \in \mathcal{B}_2$; thus, $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is continuous.

The converse to the previous statement need not be true as the following example shows: Let $(X, ^-)$ be the closure space with $X = \{a, b, c, d\}$ and with the base $\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}$ of open sets. Then the topology on the associated topological space (X, τ) is discrete. Thus, every mapping on (X, τ) is continuous. On the other hand, the mapping f on $(X, ^-)$ defined by $f(a) = a$, $f(b) = c$, $f(c) = b$, $f(d) = d$, is not continuous ($f^{-1}(\{a, b\}) = \{a, c\}$ is not open).

Proposition 6.8. *Suppose that* (X_1, P_1) *and* (X_2, P_2) *are set orthoposets,* $(X_1, ^-)$ *and* $(X_2, ^-)$ *are their associated closure spaces (with bases* P_1, P_2 *of open sets) and that* $h : P_1 \to P_2$ *is a homomorphism carried by a point mapping* $f : X_2 \to X_1$. *Then the mapping* $f : (X_2, ^-) \to (X_1, ^-)$ *is continuous.*

Proof. Suppose that $A \subset X_1$ is open. Then $A = \bigcup S$ for some $S \subset P_1$ and therefore the set $f^{-1}(A) = f^{-1}(\bigcup S) = \bigcup f^{-1}(S) = \bigcup h(S)$ is open. $\square$

Theorem 6.9. *Every automorphism of an MSR* (X, P) *is carried by a homeomorphism of* $(X, ^-)$, *where* $(X, ^-)$ *is the closure space with the base* P *of open sets.*

Proof. According to Corollary 6.6, there is a one-to-one mapping f of X onto X that carries the given automorphism h of P. It is easy to see that the isomorphism h^{-1} is carried by f^{-1} and, according to Proposition 6.8, both f and f^{-1} are continuous. $\square$

Proposition 6.10. *Every set orthoposet can be embedded into a Boolean algebra* B *such that every automorphism on it has an extension over* B.

Proof. Let (X, P) be the MSR of a given orthoposet and let B be the Boolean algebra of clopen subsets of a topological space (X, τ) with the subbase P of open sets. Suppose that h is an automorphism on P. According to Theorem 6.9, h is carried by a homeomorphism of (X, τ). It is easy to see that f carries an automorphism $\tilde{h}$ of B such that $\tilde{h}|P = h$. $\square$

Remarks 6.11. (1) In concrete orthoposets every point carries a measure, hence we can use a *generalized Stone representation* (i.e., the set representation by means of all two-valued measures) instead of an MSR. All results in this section remain valid except Corollary 6.6(1) which we have to state for orthohomomorphisms. Moreover, the embedding in Proposition 6.10 is then an orthohomomorphism.

(2) Instead of an MSR or a generalized Stone representation in Theorem 6.9 we can use for some orthoposets a set representation by means of other suitable full set of two-valued partially additive measures (cf. Corollary 6.5), e.g. by means of a full set of all two-valued Jauch-Piron measures.

7. Open problems

We have given several results—set representation of an orthoposet and extension properties of partially additive measures—for a suitable family $\mathcal{R} \subset \mathrm{OS}(P)$. The best results we have obtained for families $\mathcal{R}$ from Examples 2.2(3) and 2.2(4). These results are independent, hence it is natural to ask whether we can take the union of these families. It is not known if the set of all such defined two-valued $\mathcal{R}$-measures on an orthomodular poset is full. On the other hand, maximal $\mathcal{R}$-ideals need not be prime $\mathcal{R}$-ideals, hence we cannot use the technique given in this paper.

Further, there is an orthomodular lattice P such that the set of all two-valued $(\mathrm{OS}(B_1) \cup \mathrm{OS}(B_2))$-measures on P is not full for suitable Boolean subalgebras B_1, B_2 of P (see [19]).

Problem 7.1. Find a greater family $\mathcal{R} \subset \mathrm{OS}(P)$ for an (orthomodular) orthoposet P such that the set of all two-valued $\mathcal{R}$-measures on P is full or such that every maximal $\mathcal{R}$-ideal is a prime $\mathcal{R}$-ideal.

In Corollary 5.5 we gave a partial solution for the following problem that is interesting in the quantum logic theory (see [3]).

Problem 7.2. Which orthoposets (orthomodular posets, orthomodular lattices, resp.) can be completed to an orthomodular lattice?

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