International Journal of Theoretical Physics, Vol. 34, No. 8, 1995, 1767–1774
DOI: 10.1007/BF00676290

Subadditivity of States on Quantum Logics

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Received March 28, 1995

We give various definitions of subadditivity of states on quantum logics and present several results stating when a quantum logic with sufficiently enough "properly subadditive" states has to be (almost) a Boolean algebra.

1. INTRODUCTION

Various forms of subadditivity of states on quantum logics play important roles in quantum structures theories. The significance of Jauch–Piron states is fully accepted [see, e.g., Pták (1993) for recent results]. The importance of subadditive states is advocated, e.g., by Pulmannová and Majerník (1992) in connection with Bell inequalities.

In the present paper we give an original proof of the equivalence of some notions of subadditivity in lattice quantum logics. For partial results, alternative proofs and other notions of subadditivity see, e.g., Birkhoff (1948), Riečanová (1988), Pulmannová and Majerník (1992), Pulmannová (1993) and Pták and Pulmannová (1994).

We concentrate our attention on the following question: When does a quantum logic with sufficiently enough "properly subadditive" states have to be (almost) a Boolean algebra? This question can be restated also in the opposite way: What can we not expect from a quantum logic, should it be nonclassical?

This question has been involved in many papers; see, e.g., Rüttimann (1977), Bunce *et al.* (1985), Navara and Pták (1989), Rogalewicz (1991), Müller *et al.* (1992), Pulmannová and Majerník (1992), Pulmannová (1993) and Pták and Pulmannová (1994) for positive results and, e.g., Ovchinnikov (1993) and Müller (1993) for counterexamples.

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2. BASIC NOTIONS

Definition 2.1. A *quantum logic* is a structure $(L, \leq, ', 0, 1)$ fulfilling the following conditions:

(1) $\leq$ is a partial ordering on L with a least and a greatest element, 0, 1, respectively.
(2) $': L \to L$ is a unary mapping on L with $(a')' = a$ for any $a \in L$.
(3) If $a, b \in L$ and $a \leq b$, then $b' \leq a'$.
(4) If $a, b \in L$ and $a \leq b'$, then the supremum $a \vee b$ exists in L.
(5) If $a, b \in L$ and $a \leq b$, then there is an element $c \in L$ such that $c \leq a'$ and $b = a \vee c$ (the orthomodular law).

The operation $'$ is called the *orthocomplementation*; elements a, b of a quantum logic are called *orthogonal* (denoted by $a \perp b$) if $a \leq b'$.

The element c of condition (5) is called the *relative orthocomplement of a in b* and can be expressed as $(a \vee b')' = a' \wedge b$. In fact, the orthomodular law says that if we restrict ourselves only to elements of L less than or equal to a given $b \in L \setminus \{0\}$, then we obtain a quantum logic again.

A quantum logic is sometimes called an orthomodular poset and a *lattice quantum logic* (i.e., a quantum logic that is a lattice with respect to the given partial ordering) is sometimes called an orthomodular lattice.

Definition 2.2. A *state* on a quantum logic L is a mapping $s: L \to [0, 1]$ such that:

(1) $s(1) = 1$.
(2) $s(a \vee b) = s(a) + s(b)$ whenever $a, b \in L$ and $a \perp b$.

A *two-valued state* is a state with values in $\{0, 1\}$.

3. SUBADDITIVITY OF STATES ON LATTICE QUANTUM LOGICS

Many different notions (under various names) of subadditivity of states on lattice quantum logics have appeared in the literature. Let us add another notion and give a new proof of the equivalence of some of them.

Definition 3.1. Let L be a lattice quantum logic. A state s on L is called: *weakly subadditive* if

(1) $\quad s(a) + s(b) \geq s(a \vee b) \quad$ for any $a, b \in L$ with $a \wedge b = 0$;

subadditive if

(2) $\quad s(a) + s(b) \geq s(a \vee b) \quad$ for any $a, b \in L$;

strongly subadditive if

(3) $\quad s(a) + s(b) \geq s(a \vee b) + s(a \wedge b) \qquad$ for any $a, b \in L$;

a *valuation* if

(4) $\quad s(a) + s(b) = s(a \vee b) + s(a \wedge b) \qquad$ for any $a, b \in L$;

a *weak valuation* if

(5) $\quad s(a) + s(b) = s(a \vee b) \qquad$ for any $a, b \in L$ with $a \wedge b = 0$.

Proposition 3.2. All the notions of subadditivity of a state in Definition 3.1 are equivalent.

Proof. Let us denote by L the quantum logic in question. It is easy to see that the following implications hold: (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), (4) $\Rightarrow$ (5) $\Rightarrow$ (1). It remains to prove the following two implications:

(3) $\Rightarrow$ (4): For any $a, b \in L$ we obtain [using the condition (3) for $a', b' \in L$] $s(a)+s(b) = 1-s(a')+1-s(b') \leq 2-s(a' \vee b')-s(a' \wedge b') = s(a \wedge b)+s(a \vee b)$. Since the reverse inequality is also valid, we obtain the desired equality.

(1) $\Rightarrow$ (3): Let $a, b \in L$ and let us denote by a_1 (b_1, resp.) the relative orthocomplement of $a \wedge b$ in a (b, resp.). Since $a_1 \wedge b_1 = 0$, $(a_1 \vee b_1) \perp (a \wedge b)$ and $(a_1 \vee b_1) \vee (a \wedge b) = a \vee b$, we obtain $s(a)+s(b) = s(a_1)+s(a \wedge b)+s(b_1)+s(a \wedge b) \geq s(a_1 \vee b_1) + 2s(a \wedge b) = s(a \vee b) + s(a \wedge b)$. ■

4. SUBADDITIVITY OF STATES ON NONLATTICE QUANTUM LOGICS

The situation on nonlattice quantum logics is more complicated. While it is natural to replace $a \vee b$ by the existence of some $c \geq a, b$ in conditions (1)–(3) in Definition 3.1, this replacement in conditions (4) and (5) and an analogous replacement of $a \wedge b$ (by the existence of some $d \leq a, b$) does not seem to be a good generalization—the element d (c, resp.) is allowed to be arbitrarily small (great, resp.). It seems that, e.g., the right generalization of a valuation could be described by the following limit condition:

$$s(a) + s(b) = \inf\{s(c);\ c \in L,\ c \geq a, b\} + \sup\{s(d);\ d \in L,\ d \leq a, b\}.$$

Moreover, it might be reasonable to suppose that the above infimum and supremum are (or are arbitrarily near with the equality valid) attained.

Another approach can make use of the fact, that in the lattice case there is a suitable (from the point of view of values of states) uniform upper (lower, resp.) bound.

Definition 4.1. Let L be a quantum logic. A state s on L is called *subadditive* if the following condition

(S) there is an element $c \in L$ such that $c \geq a, b$ and $s(a) + s(b) \geq s(c)$

holds for any $a, b \in L$. Further, a state s on L is called:

Jauch–Piron if the condition (S) holds for any $a, b \in L$ with $a, b \in s^{-1}(0)$;

1-*subadditive* if the condition (S) holds for any $a, b \in L$ with $a \vee b = 1$, i.e., if $s(a) + s(b) \geq 1$ for any $a, b \in L$ with $a \vee b = 1$.

We say that a set S of (not necessarily all) states on L is a *set of uniformly subadditive states* if for any $a, b \in L$ there is an element $c \in L$ such that $c \geq a, b$ and $s(a) + s(b) \geq s(c)$ for any $s \in S$.

It is easy to see that the above definition extends Definition 3.1. The following observations will be useful in the next section.

Lemma 4.2. 1. Let s be a subadditive state on a quantum logic L. Then for any $a, b \in L$ there is a $d \in L$ with $d \leq a, b$ such that $s(a) + s(b) \leq s(d) + 1$.

2. Let s be a 1-subadditive state on a quantum logic L. Then $s(a)+s(b) \leq 1$ for any $a, b \in L$ with $a \wedge b = 0$.

Proof. 1. Let $a, b \in L$ and let us take $a', b' \in L$. According to the subadditivity of s there is a $d' \in L$ with $d' \geq a', b'$ such that $s(a') + s(b') \geq s(d')$. Thus, $d \leq a, b$ and $s(a) + s(b) = 2 - s(a') - s(b') \leq 2 - s(d') = s(d) + 1$.

2. The proof is analogous to that of part 1. ■

Lemma 4.3. Every subadditive state on a quantum logic is Jauch–Piron. Every two-valued Jauch–Piron state on a quantum logic is subadditive.

Proof. The first part is obvious. Let L be a quantum logic, s be a two-valued Jauch–Piron state on L and let $a, b \in L$. If $s(a) = s(b) = 0$ we can use the Jauch–Pironnes. If $s(a) \neq 0$ or $s(b) \neq 0$ then $s(a) + s(b) \geq 1 = s(1)$. ■

Before stating the last observation, let us define what a “sufficiently” enough states usually means.

Definition 4.4. A set S of (not necessarily all) states on a quantum logic L is called:

unital if for any $a \in L \setminus \{0\}$ there is a state $s \in S$ with $s(a) = 1$;

full if for any $a, b \in L$ with $a \not\leq b$ there is a state $s \in S$ with $s(b) < s(a)$.

It is easy to see that a full set of two-valued states is unital.

Lemma 4.5. Let S be a full set of states on a quantum logic L. Let $a, b \in L$ such that $s(a) + s(b) \leq 1$ for any $s \in S$. Then a and b are orthogonal.

Proof. Let $a, b \in L$ with $a \not\perp b$. Then $a \not\leq b'$ and there is a state $s \in S$ with $s(a) + s(b) > s(b') + s(b) = 1$. ∎

Now, let us present the main result of this section, which shows that the result of Pták and Pulmannová (1994) is a corollary of the result of Pulmannová and Majerník (1992).

Proposition 4.6. A unital set of uniformly subadditive states on a quantum logic is full.

Proof. Let L be a quantum logic with a unital set S of uniformly subadditive states and let $a \not\leq b$. According to Lemma 4.2, there is a $d \in L$ with $d \leq a, b$ such that $s(a) + s(b) \leq s(d) + 1$ for any $s \in S$. Let a_1 be the relative orthocomplement of d in a. Since $a \not\leq b$, we have $a_1 \neq 0$. Thus, there is a state $s_1 \in S$ with $s_1(a_1) = 1$. Hence, $s_1(d) = 0$, $s_1(a) = 1$ and, since $s_1(b) \leq s_1(d) + 1 - s_1(a)$, $s_1(b) = 0$. ∎

Let us note that for any pair $a, b \in L$ with $a \not\leq b$ we have found a state $s \in S$ with the property $1 = s(a) > s(b) = 0$, thus the set S is more than full.

5. WHEN A QUANTUM LOGIC HAS TO BE A BOOLEAN ALGEBRA

To prove that a quantum logic is a Boolean algebra we will make use of the following proposition.

Proposition 5.1. A quantum logic L is a Boolean algebra iff any pair of its elements is compatible, i.e., for any $a, b \in L$ there are mutually orthogonal elements $a_1, b_1, d \in L$ such that $a = a_1 \vee d$ and $b = b_1 \vee d$.

Proof. See, e.g., Pták and Pulmannová (1991). ∎

Theorem 5.2. A quantum logic with a full set of uniformly subadditive states is a Boolean algebra.

Proof. Let L be a quantum logic with a full set S of uniformly subadditive states and let $a, b \in L$. According to Lemma 4.2, there is a $d \in L$ with $d \leq a, b$ such that $s(a) + s(b) \leq s(d) + 1$ for any $s \in S$. Let us denote by a_1 (b_1, resp.) the relative orthocomplement of d in a (b, resp.). Then $s(a_1) + s(b_1) = s(a) + s(b) - 2s(d) \leq 1$ and therefore, according to Lemma 4.5, the elements a_1, b_1, d are mutually orthogonal. ∎

Now, let us give a definition of a Boolean quantum logic [for properties of Boolean quantum logics see, e.g., Tkadlec, 1993] and recall that a Boolean lattice quantum logic is a Boolean algebra.

Definition 5.3. A quantum logic L is called *Boolean* if $a \perp b$ for any pair $a, b \in L$ with $a \wedge b = 0$.

Proposition 5.4. A quantum logic with a full set of 1-subadditive states is Boolean.

Proof. Let L be a quantum logic with a full set S of 1-subadditive states. Let $a, b \in L$ with $a \wedge b = 0$. Then, according to Lemma 4.2, $s(a) + s(b) \leq 1$ for any $s \in S$. Hence, according to Lemma 4.5, a and b are orthogonal. ■

Corollary 5.5. A lattice quantum logic with a full set of 1-subadditive states is a Boolean algebra.

Theorem 5.2 and Corollary 5.5 generalize the result of Pulmannová and Majerník (1992) that was stated for a lattice quantum logic with a full set of subadditive states.

The last theorem we present here generalizes the result of Müller *et al.* (1992).

Theorem 5.6. Let L be a quantum logic and let the following conditions hold:

(1) L has a unital set of 1-subadditive states.
(2) L has a countable unital set of states.
(3) Every state on L is Jauch–Piron.

Then L is a Boolean algebra.

Before we proceed to the proof of Theorem 5.6, let us give a pair of lemmas.

Lemma 5.7. Let L be a quantum logic with a unital set of 1-subadditive states and let $a, b \in L$ be such that $a \wedge b = 0$ and $a \wedge b' = 0$. Then $a = 0$.

Proof. Let us suppose that $a \neq 0$. Then there is a 1-subadditive state s on L such that $s(a) = 1$. According to Lemma 4.2, $s(a) + s(b) \leq 1$ and $s(a) + s(b') \leq 1$. Thus, $1 = s(b \vee b') = s(b) + s(b') = 0$—a contradiction. ■

Lemma 5.8. Let L be a quantum logic such that assumptions (2) and (3) of Theorem 5.6 are fulfilled. Then for any $a, b \in L$ there is a $d \in L$ with $d \leq a, b$ such that the relative orthocomplements of d in a and in b have zero infimum (d is a maximal element of the set $\{e \in L;\ e \leq a, b\}$).

Proof. Let us denote by S a countable unital set of states on L. If $a \wedge b = 0$, we can take $d = 0$. Let us suppose that the set $L_{a,b} = \{e \in L;\ 0 < e \leq a, b\}$ is nonempty. Then, according to assumption (2), the set $S_{a,b} = \{s \in S;\ s(a) =$

$s(b) = 1\}$ is nonempty and countable. Let a state s_1 on L be a σ-convex combination (with nonzero coefficients) of all $s \in S_{a,b}$. According to assumption (3), the state s_1 is Jauch–Piron. Since $s_1(a) = s_1(b) = 1$, there is a $d \in L_{a,b}$ such that $s_1(d) = 1$. Let us show now that there is no element $e \in L_{a,b}$ that is orthogonal to d. Indeed, for any $e \in L_{a,b}$ there is a state $s_e \in S_{a,b}$ such that $s_e(e) = 1$; thus, $s_1(e) > 0$ and e and d are not orthogonal. ■

Proof. (of Theorem 5.6) Let $a, b \in L$. According to Lemma 5.8, there is a $d \in L$ with $d \leq a, b$ such that the relative orthocomplement a_1 of d in a has a zero infimum with b. According to Lemma 5.8 again, there is an $e \in L$ with $e \leq a_1, b'$ such that the relative orthocomplement a_2 of e in a_1 has a zero infimum with b'. Thus, $a_2 \wedge b = 0$ and $a_2 \wedge b' = 0$ and, according to Lemma 5.7, $a_2 = 0$. Hence, $a_1 = e$ is orthogonal to b and elements a, b are compatible. ■

As the following examples show, none of the conditions of Theorem 5.6 can be omitted.

Examples 5.9. 1. There is a countable quantum logic that it is not a Boolean algebra such that every state on it is Jauch–Piron and the set of states is unital (Ovchinnikov, 1993). The assumptions (2) and (3) of Theorem 5.6 are fulfilled.

2. There is a quantum logic such that every state on it is Jauch–Piron and the set of two-valued states is full (Müller, 1993). The assumption (3) and, according to Lemma 4.3, the assumption (1) of Theorem 5.6 are fulfilled.

3. Let X_1, X_2, X_3, X_4 be mutually disjoint countable sets and let $X = \bigcup_{i=1}^{4} X_i$. Let us put

$$\begin{aligned} L' &= \{\emptyset,\ X_1 \cup X_2,\ X_2 \cup X_3,\ X_3 \cup X_4,\ X_4 \cup X_1,\ X\}, \\ L &= \{(A \setminus F) \cup (F \setminus A);\ F \subset X \text{ is finite and } A \in L'\}. \end{aligned}$$

Then $(L, \subseteq, ^c, \emptyset, X)$, where ^{c} denotes the set-theoretic complementation in X, is a quantum logic (the axioms can be easily verified) that is not a Boolean algebra (indeed,, e.g., $X_1 \cup X_2$ and $X_2 \cup X_3$ are not compatible). Let us for any $x \in X$ define the state s_x as follows:

$$s_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases} \qquad A \in L.$$

It is easy to see that the set $\{s_x;\ x \in X\}$ is a countable unital set of two-valued Jauch–Piron states. Thus, the assumption (2) and, according to Lemma 4.3, the assumption (1) of Theorem 5.6 are fulfilled.

It should be noted that examples 5.9.1 and 5.9.2 are quite nontrivial and that other theorems of the given type can be found in Pulmannová (1993).

Let us finish with an open problem.

Problem 5.10. Find a proper definition of "subadditive state" such that the unital (full, resp.) set of such states forces a quantum logic to be a Boolean algebra.

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