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Greechie Diagrams of Small Quantum Logics with Small State Spaces

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We present Greechie diagrams of various quantum logics with small state spaces (i.e., the set of two-valued states is empty, not unital, not separating, not full, resp.). We present the smallest known examples of such so-called Kochen–Specker type constructions.

1. INTRODUCTION

Quantum logics are usually derived from Hilbert spaces where quantum propositions form an orthomodular lattice (a quantum logic) of closed subspaces. Since the 3-dimensional Hilbert space $\mathbf{R}^3$ is the least Hilbert space where the situation is nontrivial and since examples in Hilbert spaces with greater dimensions can be derived from those in $\mathbf{R}^3$, we restrict ourselves to $\mathbf{R}^3$.

Obviously, the set of states on the quantum logic $L(\mathbf{R}^3)$ of closed subspaces of $\mathbf{R}^3$ is large (full). On the other hand, there is no two-valued state on $L(\mathbf{R}^3)$. This is a consequence of the well-known Gleason theorem. While Gleason's theorem uses substantially an infinite number of elements, Kochen and Specker (1967) showed that this fact follows from a given finite number of elements (lines).

We are interested in examples of quantum logics representable in $L(\mathbf{R}^3)$ with small (empty is a special case of smallness) set of two-valued states. For a physical background (connection of two-valued states with yes–no experiments and with the hidden variable hypothesis) in this context see, e.g., Kochen and Specker (1967), Bub (1996), or Svozil and Tkadlec (1996).

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We give examples of quantum logics by means of Greechie diagrams. While a description of a set of lines by means of points on a cube surface (see, e.g., Bub, 1996) gives an insight into which lines are considered, a Greechie diagram enables better insight into why the set of two-valued states is small.

2. BASIC NOTIONS AND PROPERTIES

The basic set for our considerations is the set $L(\mathbf{R}^3)$ of closed subspaces of a 3-dimensional Hilbert space. $L(\mathbf{R}^3)$ consists of a zero subspace, lines, planes and of $\mathbf{R}^3$ and forms an orthomodular lattice (meet is the intersection, join is the span of the union, orthocomplement is the set of vectors orthogonal to all vectors in a given element).

It can be shown (see, e.g., Svozil and Tkadlec, 1996) that there is only a very limited number of types of finite subortholattices of $L(\mathbf{R}^3)$. They are either Boolean algebras (with one, two, or three atoms) or pastings of a finite number of 3-atomic Boolean algebras for a given atom—there is a line such that all other lines form orthogonal pairs orthogonal to this line (this corresponds to the 2-dimensional case). These structures are not interesting for us, hence we will use more general subsets—either sets of lines or suborthoposets.

Definition 2.1. A nonempty subset L of $L(\mathbf{R}^3)$ is called a *suborthoposet* of $L(\mathbf{R}^3)$ if

(1) $a^\perp = \{x \in \mathbf{R}^3;\ x \perp y \text{ for every } y \in a\} \in L$ whenever $a \in L$.

(2) $a \vee b = \mathrm{Sp}(a \cup b) \in L$ whenever $a, b \in L$ with $a \perp b$.

It can be shown that a suborthoposet of $L(\mathbf{R}^3)$ forms a lattice. [The lattice operation need not be the same as in $L(\mathbf{R}^3)$—the join of a pair of nonorthogonal elements might be $\mathbf{R}^3$ in a suborthoposet, while it is the plane containing these lines in $L(\mathbf{R}^3)$.] We say that a suborthoposet L of $L(\mathbf{R}^3)$ is *generated* (*orthogenerated*, resp.) by a set $M \subset L$ if every element of L can be expressed using the elements of M and the operations of join (of orthogonal elements, resp.) and orthocomplementation. Let us note that if a pair of orthogonal lines belongs to a suborthoposet of $L(\mathbf{R}^3)$, then the line orthogonal to both of them also belongs to this suborthoposet.

A *two-valued state* on a suborthoposet L of $L(\mathbf{R}^3)$ is a mapping $s : L \to \{0, 1\}$ such that $s(\mathbf{R}^3) = 1$ and $s(a \vee b) = s(a) + s(b)$ whenever $a, b \in L$ with $a \perp b$. Let us introduce the notion of a two-valued state also in another setting.

Definition 2.2. A *two-valued state* on a set $M \subset L(\mathbf{R}^3)$ of lines is a mapping $s : M \to \{0, 1\}$ such that

(1) $s(a) + s(b) \leq 1$ whenever $a, b \in M$ with $a \perp b$.

(2) $s(a) + s(b) + s(c) = 1$ whenever $a, b, c \in M$ are mutually orthogonal.

Obviously, if s is a two-valued state on a suborthoposet L of $L(\mathbf{R}^3)$, then its restriction to a set of lines in L is a two-valued state on this set.

We will use several concepts of "smallness" of the set of two-valued states: emptiness and not "large" in some of the following interpretations.

Definition 2.3. A set S of two-valued states on a suborthoposet L of $L(\mathbf{R}^3)$ is called:

(1) *unital*, if for every $a \in L$ with $a \neq \{(0,0,0)\}$ there is an $s \in S$ such that $s(a) = 1$;

(2) *separating*, if for every $a, b \in L$ with $a \neq b$ there is an $s \in S$ such that $s(a) \neq s(b)$;

(3) *full*, if for every $a, b \in L$ with $a \not\perp b$ there is an $s \in S$ such that $s(a) = s(b) = 1$.

It is well known and easy to see that a full (separating, resp.) set of two-valued states is separating (unital, resp.). All these notions are studied in quantum theories. Orthomodular posets with a full (separating, resp.) set of two-valued states are called concrete logics (partition logics, resp.); see, e.g., Pták and Pulmannová (1991), Schaller and Svozil (1994).

We will define a *unital* (*separating*, *full*, resp.) set of two-valued states on a set of lines of $L(\mathbf{R}^3)$ by the same condition as in the above definition. [The proper generalization of the notion of a separating set of two-valued states might be stronger: a unital set such that for every a, b with $a \neq b$ and $a \not\perp b$ there are two-valued states s_1, s_2 with $s_1(a) = s_1(b)$ and $s_2(a) \neq s_2(b)$.]

Every suborthoposet of $L(\mathbf{R}^3)$ can be represented by a *Greechie diagram* as follows: We represent atoms (lines in our constructions) by points and maximal subsets of mutually orthogonal atoms (triads—triples of mutually orthogonal lines in our constructions) by smooth curves (usually by line segments) containing corresponding points. We will use 'almost' Greechie diagrams—since each smooth curve connects exactly three points in our examples, we will omit points which belong to only one curve. This makes the diagrams a bit simpler.

The Greechie diagram exhibits clearly orthogonality relations. Hence, it is easy to verify whether a set of lines generates (orthogenerates, resp.) a suborthoposet of $L(\mathbf{R}^3)$—we add consecutively points which are connected by smooth curves with (which belong to the same smooth curve as, resp.) a pair of points already generated (orthogenerated, resp.). Moreover, we can easily verify whether a mapping on a set of lines is a two-valued state and properties of the state space.

3. EXAMPLES

The suborthoposet of $L(\mathbf{R}^3)$ given by the first diagram in Fig. 1 is generated by 3 lines (e.g., by those marked by a circle), orthogenerated by 6 lines

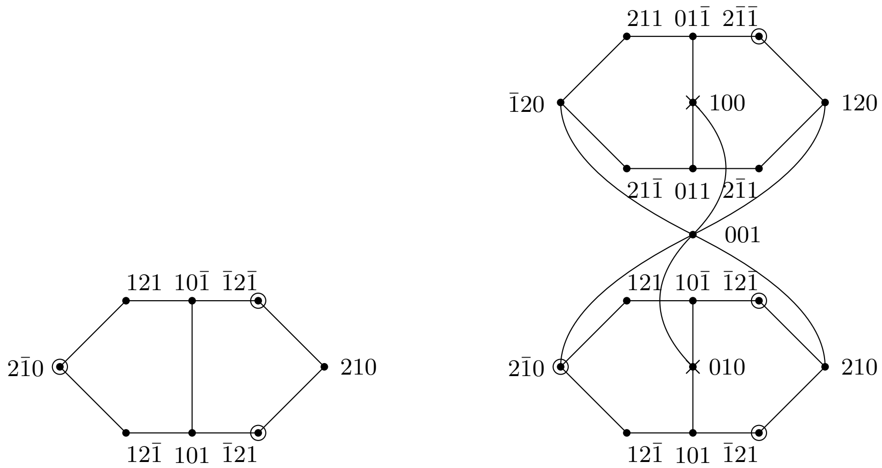

Fig. 1 'Almost' Greechie diagrams of suborthoposets of $L(\mathbf{R}^3)$ without a full and without a separating set of two-valued states [e.g., $12\bar{1} = \mathrm{Sp}(1, \sqrt{2}, -1)$].

(e.g., by all marked except $10\bar{1}$ and 101), contains 13 lines (4 of them are not drawn in slanted line segments, 1 is not drawn in the vertical line segment), 7 triads, and an 8-element set of lines [all marked; see, e.g., Kochen and Specker (1967) for both diagrams] without a full set of two-valued states. Indeed, if the set of two-valued states is full, then there is a two-valued state s such that $s(2\bar{1}0) = s(210) = 1$; hence $s(121) = s(\bar{1}2\bar{1}) = s(12\bar{1}) = s(\bar{1}21) = 0$ and therefore $s(10\bar{1}) = s(101) = 1$—a contradiction.

The suborthoposet of $L(\mathbf{R}^3)$ given by the second diagram in Fig. 1 is generated by 4 lines (e.g., by those marked by a circle), orthogenerated by 10 lines (e.g., by the same as in the previous example and by 001, 211, $2\bar{1}\bar{1}$, $21\bar{1}$), and contains 27 lines, 17 triads, and a 17-element set of lines (all marked except those which are crossed) without a separating set of two-valued states. Indeed, if $s(2\bar{1}0) = 1$ for a two-valued state s then $s(001) = s(210) = 0$ (we use the fact proven in the previous example) and therefore $s(\bar{1}20) = 1$; due to the symmetry, the reverse implication is also satisfied, hence $s(2\bar{1}0) = s(\bar{1}20)$ for every two-valued state s.

The suborthoposet of $L(\mathbf{R}^3)$ given in Fig. 2 is generated by 3 lines (e.g., by those marked by a circle), orthogenerated by 11 lines [e.g., by those given by Schütte; see Clavadetscher-Seeberger (1983)—the above-mentioned generators, vertices of the 'hexagon', 102 and 201], and contains 37 lines, 26 triads, and a 25-element set of lines (all marked) without a unital set of two-valued states. Indeed, let us suppose that there is a two-valued state on these lines such that $s(100) = 1$ and therefore $s(010) = s(001) = s(011) = s(01\bar{1}) = 0$. First, let us suppose that $s(\bar{1}02) = 1$: we consecutively obtain $s(2\bar{1}1) = s(211) = 0$, $s(11\bar{1}) =$

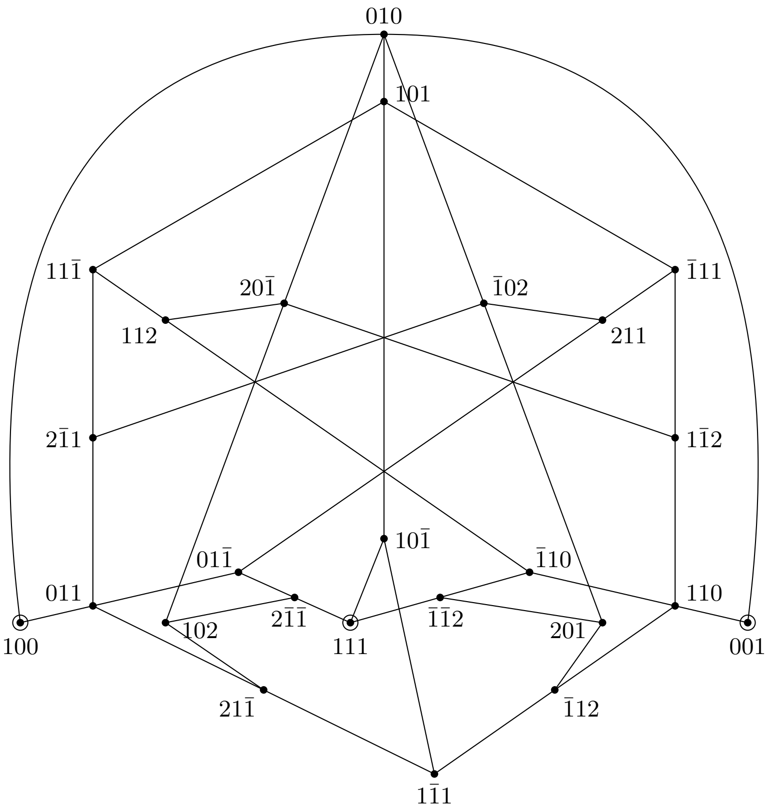

Fig. 2 'Almost' Greechie diagram of a suborthoposet of $L(\mathbf{R}^3)$ without a unital set of two-valued states [e.g., $12\bar{1} = \mathrm{Sp}(1, 2, -1)$].

$s(\bar{1}11) = 1$, $s(\bar{1}10) = s(110) = 0 = s(001)$—a contradiction. Hence $s(201) = 1$ and $s(\bar{1}12) = s(\bar{1}\bar{1}2) = 0$. Now, let us suppose that $s(102) = 1$: we obtain $s(2\bar{1}\bar{1}) = s(21\bar{1}) = 0$, $s(111) = s(1\bar{1}1) = 1$, $s(\bar{1}10) = s(110) = 0 = s(001)$—a contradiction. Hence $s(20\bar{1}) = 1$ and $s(112) = s(1\bar{1}2) = 0$. Finally, let us suppose that $s(110) = 0$: we obtain $s(\bar{1}11) = s(1\bar{1}1) = 1$, $s(101) = s(10\bar{1}) = 0 = s(010)$—a contradiction. Hence $s(\bar{1}10) = 0$ and we obtain $s(11\bar{1}) = s(111) = 1$, $s(101) = s(10\bar{1}) = 0 = s(010)$—a contradiction.

The suborthoposet of $L(\mathbf{R}^3)$ given in Fig. 3 is generated by 3 lines (e.g., by those marked by a circle), orthogenerated by 17 lines (e.g., by 100, 001, and by all lines which arise from 012, $1\bar{1}2$, $\bar{1}\bar{1}2$ using permutations of coordinates), and contains 57 lines, 40 triads, and a 33-element set of lines [all marked; given by Peres (1991)] without any two-valued state. Indeed, if there is a two-valued state, then there is a two-valued state s such that $s(010) = s(\bar{1}21) = 1$ (we use symmetries) and therefore $s(100) = s(001) = s(210) = s(0\bar{1}2) = 0$. Hence $s(\bar{1}20) = 1$, $s(21\bar{1}) = s(211) = 0$. Let us suppose that $s(120) = 1$: we obtain $s(2\bar{1}1) = s(2\bar{1}\bar{1}) = 0$, $s(011) = s(01\bar{1}) = 1$—a contradiction. Hence $s(2\bar{1}0) = 1$

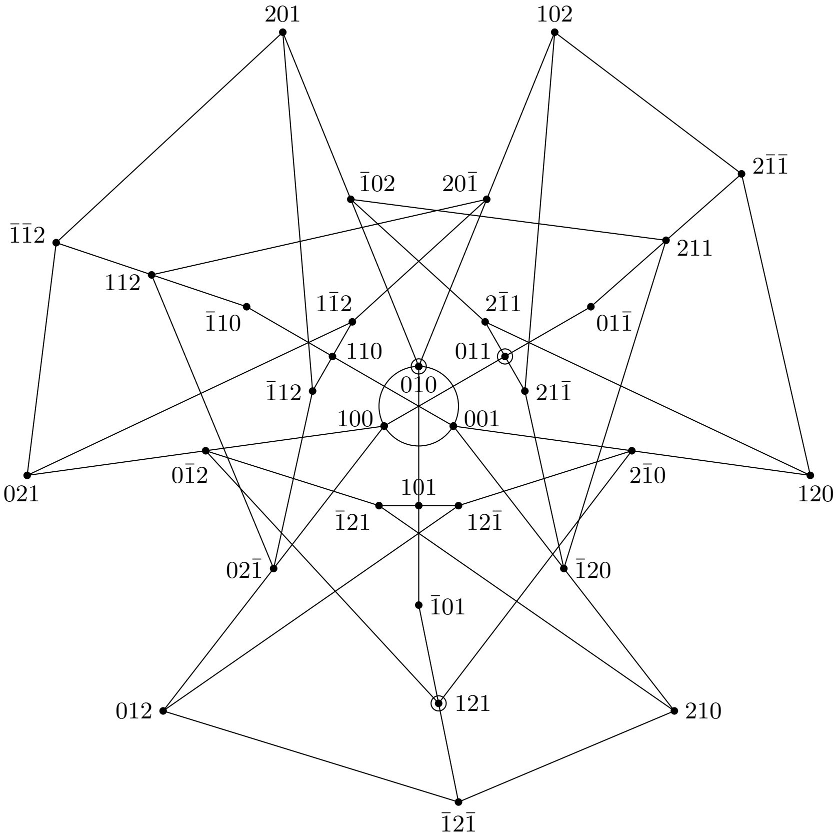

Fig. 3 'Almost' Greechie diagram of a suborthoposet of $L(\mathbf{R}^3)$ without any two-valued state [e.g., $12\bar{1} = \mathrm{Sp}(1, \sqrt{2}, -1)$].

and we obtain $s(121) = 0$, $s(\bar{1}2\bar{1}) = 1$, $s(012) = 0$, $s(02\bar{1}) = 1$, $s(\bar{1}12) = s(112) = 0$. Since $s(021) = 1$ we obtain $s(\bar{1}\bar{1}2) = s(1\bar{1}2) = 0$, $s(\bar{1}10) = s(110) = 1$—a contradiction.

It should be noted that there is also another example of 33 lines without any two-valued state by Bub (1996) and an example with 31 lines by Conway and Kochen [unpublished; see Bub (1996)].

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