Acta Polytechnica **36** (1996), 5–10

THE DOG-AND-RABBIT CHASE REVISITED

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Keywords: Differential equation

We consider a generalized setup of the standard problem of a dog chasing a rabbit. We find an explicit description of the dog's path and, as a by-product, we establish the formula for the duration of the chase (provided that the chase is terminated). Also, we find the solution for the limit distance problem in the case when the dog's velocity equals the rabbit's velocity.

1. Formulation of the Problem

Let us consider a rabbit running in a straight line at constant speed, v_{r} ($v_{\mathrm{r}} > 0$), while a dog runs at constant speed, v_{d} ($v_{\mathrm{d}} > 0$), heading at each moment directly towards the rabbit. We want to work out the function that describes the dog's path of pursuit, and we also want to find how long the chase lasts. If the speeds of the dog and the rabbit are equal, we want to determine their limit distance.

Obviously, the problem we have posed also admits alternative formulations. A modern formulation might involve electronic games and fighter planes that chase each other. (This formulation may soon become prevalent as the realm of animals shrinks for the younger generation ...) Anyway, such problems have proved to be of considerable pedagogical value in advanced calculus [1, 2]. The solutions however are usually achieved in an implicit form, or they involve quite elaborate maneuvers with polar coordinates. In this note we present an explicit solution that uses elementary theory of differential equations only.

2. Solution

Let us consider Figures 1–4. They indicate typical situations of the chase and help set up the mathematical formulation of the problem. Let us first consider Fig. 1.

Let the point $\mathrm{D}_0 = [d, 0]$ on the x-axis symbolize the position of the dog at the beginning of the chase and $\mathrm{R}_0 = [0, r]$ the position of the rabbit. We suppose that the rabbit runs up the y-axis. We also suppose that $d > 0$ (if $d = 0$ the problem trivializes). Let us now derive the function $y(x)$ of the dog's pursuit ($x \in (0, d\rangle$). Considering the triangle RYD in Fig. 1 and recalling that the integral formula $\int_x^d \sqrt{1 + y'(s)^2}\,\mathrm{d}s$ means the length of the dog's path from D_0 to D, we obtain for any $x \in (0, d)$

$$-y'(x) = \frac{r + v_{\mathrm{r}} \cdot \frac{1}{v_{\mathrm{d}}} \int_x^d \sqrt{1 + y'(s)^2}\,\mathrm{d}s - y(x)}{x}.$$

After rearranging, we get

$$-x \cdot y'(x) = r - y(x) + \frac{v_{\mathrm{r}}}{v_{\mathrm{d}}} \int_x^d \sqrt{1 + y'(s)^2}\,\mathrm{d}s.$$

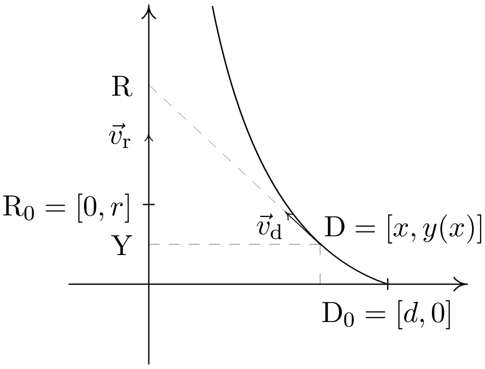

Fig. 1: $v_{\mathrm{d}} < v_{\mathrm{r}}$, $r > 0$

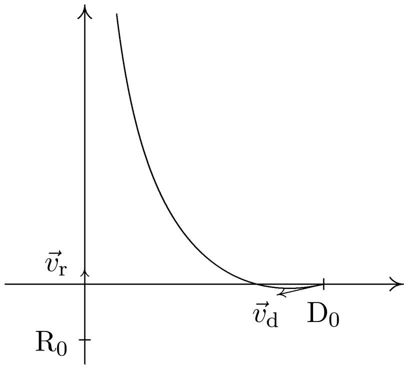

Fig. 2: $v_{\mathrm{d}} < v_{\mathrm{r}}$, $r < 0$

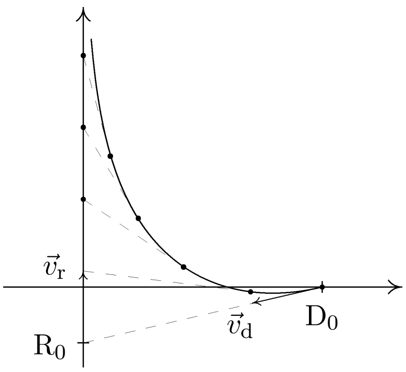

Fig. 3: $v_{\mathrm{d}} = v_{\mathrm{r}}$

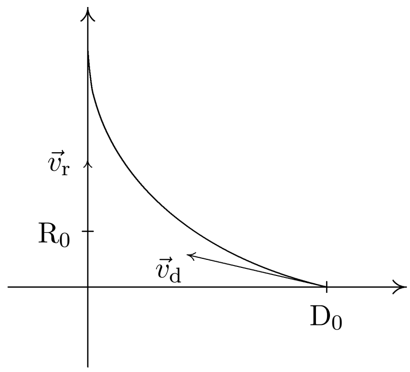

Fig. 4: $v_{\mathrm{d}} > v_{\mathrm{r}}$

The differentiation of the latter equality gives

$$-x \cdot y''(x) - y'(x) = -y'(x) - \frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\sqrt{1 + y'(x)^2}.$$

From this formula we obtain the desired differential equation for $y(x)$:

$$y''(x) = \frac{v_{\mathrm{r}}}{v_{\mathrm{d}} x}\sqrt{1 + y'(x)^2}.$$

The initial conditions are $y(d) = 0$ and $y'(d) = -r/d$.

We shall now find the solution to the latter differential equation. Putting $y'(x) = u(x)$, we transform our equation into the form

$$u'(x) = \frac{v_{\mathrm{r}}}{v_{\mathrm{d}} x}\sqrt{1 + u(x)^2}, \qquad u(d) = -\frac{r}{d}.$$

Separating, we obtain

$$\frac{\mathrm{d}u}{\sqrt{1 + u(x)^2}} = \frac{v_{\mathrm{r}}}{v_{\mathrm{d}}} \cdot \frac{\mathrm{d}x}{x}$$

and therefore

$$\operatorname{argsinh} u(x) = \frac{v_{\mathrm{r}}}{v_{\mathrm{d}}} \cdot \log x + \log C = \log C x^{v_{\mathrm{r}}/v_{\mathrm{d}}} \qquad (C > 0).$$

This means

$$u(x) = y'(x) = \sinh \log C x^{v_{\mathrm{r}}/v_{\mathrm{d}}} = \frac{1}{2}\left(C x^{v_{\mathrm{r}}/v_{\mathrm{d}}} - \frac{1}{C} x^{-v_{\mathrm{r}}/v_{\mathrm{d}}}\right).$$

We now have to specify the constant C ($C > 0$). Considering the initial value condition $y'(d) = -r/d$, we obtain

$$\frac{1}{2}\left(C d^{v_{\mathrm{r}}/v_{\mathrm{d}}} - \frac{1}{C} d^{-v_{\mathrm{r}}/v_{\mathrm{d}}}\right) = -\frac{r}{d}.$$

This gives

$$d^{2v_{\mathrm{r}}/v_{\mathrm{d}}} C^2 + 2\frac{r}{d} d^{v_{\mathrm{r}}/v_{\mathrm{d}}} C - 1 = 0.$$

Hence,

$$C = \frac{\sqrt{r^2 + d^2} - r}{d} \cdot d^{-v_{\mathrm{r}}/v_{\mathrm{d}}}$$

in view of our assumption of $C > 0$. By simple algebra,

$$\frac{1}{C} = \frac{\sqrt{r^2 + d^2} + r}{d} \cdot d^{v_{\mathrm{r}}/v_{\mathrm{d}}}.$$

Finally, we obtain

$$y'(x) = \frac{1}{2}\left(\frac{\sqrt{r^2 + d^2} - r}{d}\left(\frac{x}{d}\right)^{v_{\mathrm{r}}/v_{\mathrm{d}}} - \frac{\sqrt{r^2 + d^2} + r}{d}\left(\frac{x}{d}\right)^{-v_{\mathrm{r}}/v_{\mathrm{d}}}\right).$$

In integrating the latter equation, we have to distinguish two cases.

1. Suppose that $v_{\mathrm{d}} \neq v_{\mathrm{r}}$. Then

$$y(x) = \frac{1}{2}\left(\frac{\sqrt{r^2 + d^2} - r}{1 + \frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}} \cdot \left(\frac{x}{d}\right)^{1 + v_{\mathrm{r}}/v_{\mathrm{d}}} - \frac{\sqrt{r^2 + d^2} + r}{1 - \frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}} \cdot \left(\frac{x}{d}\right)^{1 - v_{\mathrm{r}}/v_{\mathrm{d}}}\right) + B,$$

where B is a constant which is determined by the initial condition $y(d) = 0$:

$$B = -\frac{1}{2}\left(\frac{\sqrt{r^2 + d^2} - r}{1 + \frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}} - \frac{\sqrt{r^2 + d^2} + r}{1 - \frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}}\right) = \frac{\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\sqrt{r^2 + d^2} + r}{1 - \left(\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\right)^2}.$$

Hence,

$$y(x) = \frac{1}{2}\left(\frac{\sqrt{r^2 + d^2} - r}{1 + \frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}} \cdot \left(\frac{x}{d}\right)^{1 + v_{\mathrm{r}}/v_{\mathrm{d}}} - \frac{\sqrt{r^2 + d^2} + r}{1 - \frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}} \cdot \left(\frac{x}{d}\right)^{1 - v_{\mathrm{r}}/v_{\mathrm{d}}}\right) + \frac{\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\sqrt{r^2 + d^2} + r}{1 - \left(\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\right)^2}.$$

If $v_{\mathrm{d}} < v_{\mathrm{r}}$, then $\lim_{x \to 0_+} y(x) = +\infty$ and therefore the dog does not catch the rabbit.

If $v_{\mathrm{d}} > v_{\mathrm{r}}$, then

$$\lim_{x\to 0_+} y(x) = \frac{\dfrac{v_{\mathrm{r}}}{v_{\mathrm{d}}} \cdot \sqrt{r^2+d^2} + r}{1 - \left(\dfrac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\right)^2}.$$

This means that the time $t^\dagger$ of the duration of the chase computes as follows:

$$t^\dagger = \left(\frac{\dfrac{v_{\mathrm{r}}}{v_{\mathrm{d}}} \cdot \sqrt{r^2+d^2} + r}{1 - \left(\dfrac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\right)^2} - r \right) \cdot \frac{1}{v_{\mathrm{r}}} = \frac{\sqrt{r^2+d^2} + \dfrac{v_{\mathrm{r}}}{v_{\mathrm{d}}} \cdot r}{\left(1 - \left(\dfrac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\right)^2\right) \cdot v_{\mathrm{d}}}.$$

As a corollary, we see that the length d^* of the total dog's run is

$$d^* = \frac{\sqrt{r^2+d^2} + \dfrac{v_{\mathrm{r}}}{v_{\mathrm{d}}} \cdot r}{1 - \left(\dfrac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\right)^2}$$

and the length $d^\dagger$ of the total rabbit's run is $d^\dagger = (v_{\mathrm{r}}/v_{\mathrm{d}}) \cdot d^*$.

2. Suppose that $v_{\mathrm{d}} = v_{\mathrm{r}}$. Then the integrating of the equation for $y'(x)$ gives

$$y(x) = \frac{1}{2} \left(\frac{\sqrt{r^2+d^2} - r}{2} \cdot \frac{x^2}{d^2} - \left(\sqrt{r^2+d^2} + r\right) \cdot \log x \right) + K,$$

where K is a constant which is determined by the initial condition $y(d) = 0$:

$$K = -\frac{1}{2} \left(\frac{\sqrt{r^2+d^2} - r}{2} - \left(\sqrt{r^2+d^2} + r\right) \cdot \log d \right).$$

Hence,

$$y(x) = -\frac{\sqrt{r^2+d^2} + r}{2} \cdot \log \frac{x}{d} - \frac{\sqrt{r^2+d^2} - r}{4} \cdot \left(1 - \frac{x^2}{d^2}\right).$$

We therefore see that $\lim_{x\to 0_+} y(x) = +\infty$ and hence, logically enough, the dog does not catch the rabbit. In this case however the distance between the dog and the rabbit should converge to a finite value. We shall now find this value. For a given x ($x \in \langle 0, d \rangle$), the dog finds himself at the point $[x, y(x)]$. Considering again the triangle RYD in Fig. 1, we see that the distance $d(x)$ between the dog and the rabbit is

$$\begin{aligned} d(x) &= \sqrt{x^2 + \left(r + v_{\mathrm{r}} \cdot \frac{1}{v_{\mathrm{d}}} \int_x^d \sqrt{1 + y'(s)^2}\, \mathrm{d}s - y(x) \right)^2} \\ &= \sqrt{x^2 + (-x \cdot y'(x))^2}. \end{aligned}$$

We want to compute $\lim_{x\to 0_+} d(x)$. Since $v_{\mathrm{r}}/v_{\mathrm{d}} = 1$ in the case under consideration, we have

$$y'(x) = \frac{1}{2} \left(\frac{\sqrt{r^2+d^2} - r}{d^2} \cdot x - \frac{\sqrt{r^2+d^2} + r}{x} \right).$$

Substituting now for $y'(x)$, we consecutively obtain

$$\begin{aligned}\lim_{x\to 0_+} d(x) &= \lim_{x\to 0_+} |-x\cdot y'(x)| \\ &= \lim_{x\to 0+} \frac{1}{2}\left(-\frac{\sqrt{r^2+d^2}-r}{d^2}\cdot x^2+\sqrt{r^2+d^2}+r\right) \\ &= \frac{\sqrt{r^2+d^2}+r}{2}.\end{aligned}$$

3. Summary

Let us finally summarize our results. Let $y(x)$ stand for the function whose graph is the dog's path ($x\in(0,d\rangle$).

1. If $v_{\mathrm{d}} < v_{\mathrm{r}}$ (see Figs. 1, 2), then

$$y(x)=\frac{1}{2}\left(\frac{\sqrt{r^2+d^2}-r}{1+\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}}\cdot\left(\frac{x}{d}\right)^{1+v_{\mathrm{r}}/v_{\mathrm{d}}}-\frac{\sqrt{r^2+d^2}+r}{1-\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}}\cdot\left(\frac{x}{d}\right)^{1-v_{\mathrm{r}}/v_{\mathrm{d}}}\right)+\frac{\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\sqrt{r^2+d^2}+r}{1-\left(\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\right)^2}.$$

   The chase does not terminate.

2. If $v_{\mathrm{d}} = v_{\mathrm{r}}$ (see Fig. 3), then

$$y(x)=-\frac{\sqrt{r^2+d^2}+r}{2}\cdot\log\frac{x}{d}-\frac{\sqrt{r^2+d^2}-r}{4}\cdot\left(1-\frac{x^2}{d^2}\right).$$

   The chase does not terminate but the distance between the dog and the rabbit converges to the number

$$\frac{\sqrt{r^2+d^2}+r}{2}.$$

3. Finally, if $v_{\mathrm{d}} > v_{\mathrm{r}}$ (see Fig 4), then again

$$y(x)=\frac{1}{2}\left(\frac{\sqrt{r^2+d^2}-r}{1+\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}}\cdot\left(\frac{x}{d}\right)^{1+v_{\mathrm{r}}/v_{\mathrm{d}}}-\frac{\sqrt{r^2+d^2}+r}{1-\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}}\cdot\left(\frac{x}{d}\right)^{1-v_{\mathrm{r}}/v_{\mathrm{d}}}\right)+\frac{\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\sqrt{r^2+d^2}+r}{1-\left(\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\right)^2}.$$

   The chase then takes the time $t^\dagger$, where

$$t^\dagger=\frac{\sqrt{r^2+d^2}+\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\cdot r}{\left(1-\left(\frac{v_{\mathrm{r}}}{v_{\mathrm{d}}}\right)^2\right)\cdot v_{\mathrm{d}}}.$$

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Ještě jednou jak pes honí zajíce

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Uvažujeme obecný případ standardního problému jak popsat trajektorii psa honícího zajíce. Nacházíme explicitní popis této trajektorie a výraz pro dobu lovu (pokud je tento lov úspěšný). Řešíme též otázku limitní vzdálenosti psa a zajíce v případě, že jejich rychlosti jsou stejné.

Received: 1995. 09. 04

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