Tatra Mountains Math. Publ. **3** (1993), 155–160

BOOLEAN ORTHOPOSETS AND TWO-VALUED JAUCH–PIRON STATES

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ABSTRACT. A Boolean orthoposet is the orthoposet P fulfilling the following condition: if $a \wedge b = 0$ then $a \perp b$. Boolean orthoposets enjoy some properties of Boolean algebras but, in some sense, they are far away of them [3, 4, 8]. An important notion in quantum logic theory is the notion of Jauch–Piron state [2, 5, 6, 7]. In this paper, we show connections between Boolean orthoposets and orthoposets with a proper number of Jauch–Piron states: We demonstrate that an orthoposet with "enough" two-valued Jauch–Piron states is Boolean and, on the other hand, we give several examples of Boolean orthoposets without any two-valued Jauch–Piron state.

1. Basic notions

The connections mentioned above can be studied for orthoposets without assuming the orthomodular law and even without assuming the existence of suprema of pairs of orthogonal elements. Since the main purpose of this paper is to present examples of Boolean orthomodular orthoposets without any two-valued Jauch–Piron state, it seems to be more convenient to deal with more familiar (and less general) notions. The details concerning a general setting can be found in [9].

Definition 1.1. An *orthomodular poset* is a triple $(P, \leq, ')$ such that

(1) $(P, \leq)$ is a partially ordered set with a least and a greatest elements 0, 1;

(2) $' : P \to P$ is an *orthocomplementation*, i.e., (i) $a'' = a$, (ii) $a \leq b \Rightarrow b' \leq a'$, (iii) $a \vee a' = 1$ for every $a, b \in P$;

(3) the *orthomodular law* is valid in $(P, \leq, ')$, i.e., (i) $a \vee b \in P$ whenever $a, b \in P$ are *orthogonal* (i.e., $a \leq b'$, denoted by $a \perp b$), (ii) $b = a \vee (b \wedge a')$ whenever $a \leq b$.

An orthomodular poset $(P, \leq, ')$ is *Boolean* if

(4) $a \perp b$ whenever $a \wedge b = 0$.

AMS Subject Classification (1991): 03G12, 81P10.

Key words: orthoposet, boolean orthoposet, Jauch–Piron state.

The author gratefully acknowledges the support of the grant no. 201/93/0953 of Grant Agency of the Czech Republic.

Let us note that the property $a \vee a' = 1$ of the orthocomplementation is a consequence of the orthomodular law. In the sequel we shall shortly write P instead of $(P, \leq, ')$.

Definition 1.2. A *two-valued Jauch–Piron state* on an orthomodular poset P is a mapping $s : P \to \{0, 1\}$ such that

(1) $s(1) = 1$;
(2) $s(a \vee b) = s(a) + s(b)$ whenever $a \perp b$;
(3) $s(a) = s(b) = 1$ implies that there is a $c \leq a, b$ with $s(c) = 1$.

Let us note that the condition (3) can be reformulated dually for a finite number of elements:

(3') $s(a_1) = \ldots = s(a_n) = 0$ implies that there is a $c \geq a_1, \ldots, a_n$ with $s(c) = 0$.

2. Concrete representation

Every Boolean orthomodular poset has a suitable set representation.

Definition 2.1. An orthomodular poset P is called *concrete* if there is a set X (called *domain of* P) such that $P \subset \exp X$ and such that

(1) $0 = \emptyset$;
(2) $A \leq B$ iff $A \subset B$;
(3) $A' = X \setminus A$.

Let us observe that $A \perp B$ iff $A \cap B = \emptyset$ and that $1 = X$ for every concrete orthomodular poset P with the domain X. It can be easily checked that a family $P \subset \exp X$ forms a concrete orthomodular poset iff the following conditions are fulfilled:

(1) $\emptyset \in P$;
(2) $X \setminus A \in P$ whenever $A \in P$;
(3) $A \cup B \in P$ whenever $A, B \in P$ with $A \cap B = \emptyset$.

Thus, starting with an arbitrary family $P' \subset \exp X$ we can construct the least concrete orthomodular poset $P \supset P'$ with the domain X (we say that P is *generated* by P') — every nonempty element of P arises from elements of P' as a result of a finite number of operations of relative complement and of union of two disjoint elements.

The following theorem was proved in [4].

Theorem 2.2. *Every Boolean orthomodular poset has a concrete representation.*

According to this theorem we can (and we shall) use this concrete representation in the sequel. A concrete orthomodular poset P is Boolean iff the following condition (reformulation of condition (4) of Definition 1.1) is fulfilled:

(B) $A \cap B \neq \emptyset$ (i.e., there is an $x \in A \cap B$) implies that there is a $C \in P \setminus \{\emptyset\}$ with $C \subset A \cap B$ (see Fig. 1).

Let us note that there need not be any connection between the point x and the element C.

Every point $x \in X$ corresponds to some state s_x (*carried by a point* x) on a concrete orthomodular poset with the domain X defined by

$$s_x(E) = \begin{cases} 0, & \text{if } x \notin E, \\ 1, & \text{if } x \in E. \end{cases}$$

The state s_x is Jauch–Piron iff the following condition (reformulation of condition (3) of Definition 1.2) is fulfilled:

(JP) $x \in A \cap B$ implies that there is a $C \in P$ with $x \in C \subset A \cap B$ (see Fig. 2).

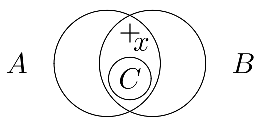

Fig. 1

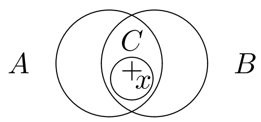

Fig. 2

It is easily seen that conditions (B) and (JP) are closely related. In order to have a concrete representation of an orthomodular poset P, it is necessary (and sufficient) to have a *full* set S of two-valued states on P (for every pair $a \not\leq b$ there is an $s \in S$ with $s(a) \not\leq s(b)$) (see e.g. [1, Theorem 3.28] or [6, Theorem 2.2.1]). Thus, comparing conditions (B) and (JP) we obtain:

Theorem 2.3. *Every orthomodular poset with a full set of two-valued Jauch–Piron states (i.e., with such a concrete representation that every point of the domain carries a Jauch–Piron state) is Boolean.*

Let us state a lemma we shall use in the sequel.

Lemma 2.4. *Let s be a two-valued Jauch–Piron state on a concrete orthomodular poset P. Let $B \in P$ with $s(B) = 1$ and let $A_1, \ldots, A_n \in P$ such that $B \subset A_1 \cup \ldots \cup A_n$. Then there is an $A \in \{A_1, \ldots, A_n\}$ with $s(A) = 1$.*

Proof. Let us suppose that $s(A_1) = \ldots = s(A_n) = 0$ and seek a contradiction. According to the reformulation (3') of condition (3) of Definition 1.2, there is a $C \in P$ with $s(C) = 0$ such that $C \supset A_1 \cup \ldots \cup A_n$. Since $B \subset A_1 \cup \ldots \cup A_n \subset C$ we have $s(B) \leq s(C) = 0$ which is a contradiction. □

3. Ideas of construction

We shall give examples which prove the following statement that completes Theorem 2.3.

Theorem 3.1. *There is a Boolean orthomodular poset that has no two-valued Jauch–Piron state.*

Before we construct an appropriate orthomodular poset P, let us give the following ideas and conditions with a sketch of the proof.

(1) P is a concrete orthomodular poset generated by a "nice" subsets of some $X \subset \mathbf{R}^n$. Let "nice" set means convex and open with some points of its boundary.

(2) Conditions that force P to be Boolean:

(2a) For every $A, B \in P$ with $A \cap B \neq \emptyset$ there is a nonempty open subset of $A \cap B$.

(2b) Every nonempty open subset of X contains some nonempty element of P.

Condition (2a) is obviously fulfilled for open sets A, B. Since usually not all elements of P can be open, every element $E \in P$ will include some (appropriately chosen) points of its boundary. To fulfill condition (2b) we shall use at least one bounded generator such that any homothetic image of it belonging to X is a generator, too.

If the above mentioned conditions are fulfilled, then for a two-valued Jauch–Piron state s on P with $s(A_1) = 1$ for some bounded set A_1 we can use the following procedure: A_1 can be covered by a finite number of "smaller" elements of P. According to Lemma 2.4, there is an element of this covering with $s(A_2) = 1$. Repeating this procedure we (can) obtain a sequence $A_1, A_2, \ldots \in P$ that converges to some x with the following property (we say that the state s is *concentrated at* x):

(C) $s(E) = 0$ whenever $E \in P$ with $\mathrm{dist}(E, x) > 0$.

(3) Conditions that force P to have no two-valued Jauch–Piron state.

(3a) Density of every generator of P at every $x \in X$ is a multiple of $1/K$ (for a suitable positive integer K).
Density of a set E at a point x is defined by

$$D(E, x) = \lim_{r \to 0_+} \frac{\lambda(E \cap B(x, r))}{\lambda(B(x, r))}$$

where $B(x, r)$ is the ball with the center x and the radius r, λ denotes the n-dimensional Lebesgue measure.

(3b) For every Jauch–Piron state s on P concentrated at x there are $A, B \in P$ such that $s(A) = s(B) = 1$ and $D(A \cap B, x) < 1/K$.

We have shown that every two-valued Jauch–Piron state is concentrated at some x. According to condition (3b), for every $C \subset A \cap B$ we have $D(C, x) \leq D(A \cap B, x) < 1/K$. Since condition (3a) is fulfilled for any $E \in P$ (this condition is preserved by forming unions and relative complements) we have $D(C, x) = 0$. Hence, according to condition (1), $\mathrm{dist}(C, x) > 0$ and, according to condition (C), $s(C) = 0$. Thus, s is not Jauch–Piron.

4. Examples

Let us give several examples of Boolean orthomodular posets without any two-valued Jauch–Piron state.

Example 4.1. Let X be the square (in the plane $\mathbf{R}^2$, see Fig. 3). And let P be generated by bilateral rectangular triangles $E \subset X$ fulfilling the following condition:

> a point x of the boundary of E belongs to E iff x is on the "left boundary" (i.e., E is placed to the right from x) or on the "bottom boundary" (i.e., E is placed above x) for horizontal boundary line segments (see Fig. 4 for examples.).

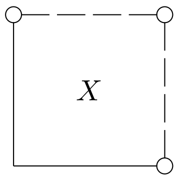

Fig. 3

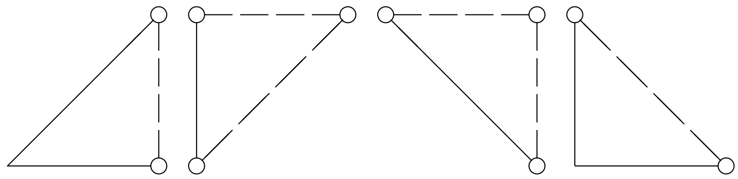

Fig. 4

It is easy to see that X and all generators have the following property of E:

(∗) $x \in E$ iff x is a "left bottom" boundary for some open angle starting with horizontal halfline that "locally belongs" to E ($A(x, \alpha)$ in Fig. 5).

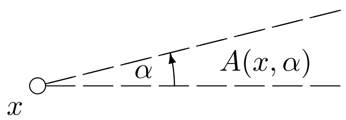

Fig. 5

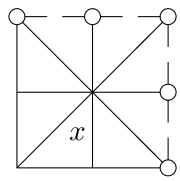

Fig. 6

Since this property is preserved by forming unions and relative complements (boundaries are finite unions of line segments), every $E \in P$ fulfills it. Since this property is preserved by forming intersections, condition (2a) is fulfilled. It is easy to see that conditions (1), (2b), (3a) are fulfilled ($K = 8$ in condition (3a)). It remains to check condition (3b). Indeed, we can cover the neighbourhood of x (in X) by 8 (4 or 2 if x is on the boundary of X) mutually disjoint generators (see Fig. 6). Since s is concentrated at x, there is one of these generators, say A, with $S(A) = 1$. Analogously, we obtain a $B \in P$ with $s(B) = 1$ if we rotate the whole situation around x by $2\pi/16$. (If x is on the boundary of X, one triangle will not be a subset of X — instead of it we take the orthocomplement of the union of remaining ones.) Then $D(A \cap B, x) = 1/16 < 1/8$.

Let us present further examples without proof (it is analogous to the previous one). A larger example we obtain if we use density property of generators instead of their description.

Example 4.2. Let X be the square as in Example 4.1 and let P be the concrete orthomodular poset of all $E \subset X$ such that the boundary of E consists of a finite number of line segments, condition $(*)$ from Example 4.1 is fulfilled and such that $D(E, x)$ is a multiple of $1/8$ for every $x \in \mathbf{R}^2$.

Another example we obtain if we use open balls.

Example 4.3. Let P be the orthomodular poset generated by open balls and by some open half-spaces (e.g., for every pair of opposite orientations of hyperplanes we choose exactly one and for every hyperplane we take the open half-space with the chosen orientation) in $\mathbf{R}^n$, $n \geq 3$.

Let us note that if we omit open half-spaces as generators, the resulting orthomodular poset will have one two-valued Jauch–Piron state that takes the value 0 exactly on bounded sets.

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Received April 22, 1993

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