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THE SOLUTION TO A PROBLEM POSED BY P. KONÔPKA

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ABSTRACT. We present an example of an atomistic set-representable orthomodular poset L that do not fulfill the condition: "For any maximal set M of mutually noncompatible atoms there is a two-valued state s on L such that $s|_M = 1$". This answers a problem posed by P. Konôpka. We compare the above condition with another condition presented by P. Konôpka at this Conference: "Any maximal set of mutually noncompatible atoms intersects every block".

Example 1. Let $X = \{1,2,3,4,5,6\}$ and let $L = \{A \subset X;\ |A| \text{ is even}\}$ with the ordering by inclusion and the orthocomplementation given by the set-theoretic complementation in X. Then L is an atomistic orthomodular poset and $M = \big\{\{1,2\},\{2,3\},\{3,1\}\big\}$ is a (maximal) set of mutually noncompatible atoms in L such that there is no state s on L with $s|_M = 1$. Indeed, if s is a state on L with $s|_M = 1$ then $s|_P = 0$ for the finite partition $P = \big\{\{3,6\},\{1,4\},\{2,5\}\big\}$ of unity in L — a contradiction.

The following example is smaller but needs better knowledge of quantum logics.

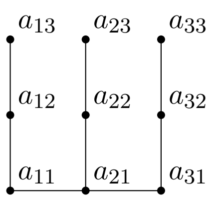

Fig. 1

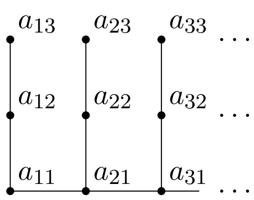

Fig. 2

Example 2. Let L be the orthomodular lattice given by the Greechie diagram on Fig. 1. Then L is atomistic and $M = \{a_{13}, a_{23}, a_{33}\}$ is a maximal set of mutually noncompatible atoms such that there is no state s on L with $s|_M = 1$. Indeed, if s is a state on L with $s|_M = 1$ then $s|_P = 0$ for the finite partition $P = \{a_{11}, a_{21}, a_{31}\}$ of unity in L — a contradiction.

It remains to prove that L is set-representable. It suffices to find for any pair a_{ij}, a_{kl} $(i,j,k,l \in \{1,2,3\})$ of noncompatible atoms (i.e., $i \neq k$ and $\{j,l\} \neq$

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$\{1\}$) a two-valued state s on L such that $s(a_{ij}) = s(a_{kl}) = 1$. It is easy to see that such a state can be defined by (A denotes the set of atoms in L)

$$s_{ijkl}^{-1}(1) \cap A = \{a_{ij}, a_{kl}, a_{m3}\}, \qquad m \neq i, k.$$

It should be noted that the above examples do not fulfill also the following condition presented by P. Konôpka at this Conference (see also [1]): "Any maximal set of mutually noncompatible atoms intersects every block (= maximal Boolean subalgebra)".

In fact, this is a stronger condition in this context. If L is an atomistic orthomodular poset with a maximal set M of mutually noncompatible atoms that intersects every block in L, then the condition (A denotes the set of atoms in L)

$$s^{-1}(1) \cap A = M$$

defines a (completely additive) two-valued state s on L with $s|_M = 1$. The rest follows from the following example.

EXAMPLE 3. Let L be the orthomodular lattice given by the Greechie diagram on Fig. 2—the set of atoms in L is $A = \big\{a_{ij};\ i \in I,\ j \in \{1,2,3\}\big\}$, where $I \supset \{1,2,3\}$ is an infinite index set. Then L is atomistic and $M = \{a_{i3};\ i \in I\}$ is a maximal set of mutually noncompatible atoms in L that does not intersect the block containing $\{a_{i1};\ i \in I\}$.

On the other hand, for any maximal set M of mutually noncompatible atoms in L there is a two-valued state s on L such that $s|_M = 1$. Indeed, if $M \cap \{a_{i1};\ i \in I\} \neq \emptyset$ then M intersects every block of L and the state s is defined by $s^{-1}(1) \cap A = M$. If $M \cap \{a_{i1};\ i \in I\} = \emptyset$, then $s^{-1}(1) \cap A = M$ again and it remains to define the state s on the block containing $\{a_{i1};\ i \in I\}$. It can be done as follows:

$$s\Big(\bigvee_{i\in J} a_{i1}\Big) = \begin{cases} 1, & \text{for } J \subset I \text{ infinite,} \\ 0, & \text{otherwise.} \end{cases}$$

Remark. If I and J in the above Example are uncountable sets, we obtain an example for σ-additive states (etc.).

REFERENCES

[1] KONÔPKA, P.: *Generalization of quantum logics and sets with relative inverses by equational characterization for partial algebras*, Tatra Mt. Math. Publ. **10** (1997), 183–197.

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