This talk has been very much influenced by Bob Coecke’s views on QM.

Thank you, Bob.
I am not a physicist. That’s bad…
I am a category theorist. So what?

Hear, hear:

*We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance.*


Wow! That’s a bit depressing…

**No, it isn’t! CT means doing physics all the time!**
What does Category Theory Bring to Quantum Physics and Quantum Computing?

In both physics and computing (and everyday life for that matter):

1. We *manipulate data*. These data have various *types*.
2. We can *concatenate* these manipulations: both *sequentially* and *in parallel*.

The above is essentially what category theory is about!
Intro
Forget the Matrices, It’s all Wires and Boxes
Classical vs Quantum

Quantum Teleportation
Superdense Coding

Pictorial Notation

Category Theory

\[
\begin{array}{c}
B \\
f \\
A
\end{array}
\]

Picture Calculus

\[
\begin{array}{c}
B \\
f \\
A
\end{array}
\]

Intuition: \( A, B \) are the state spaces, \( f \) is the transformation.
So it’s wires and boxes (plus axioms — later) instead of vector spaces, matrices, linear transformations, etc.
### Picture Calculi in Physics and Category Theory

Quantum Teleportation

Introduced in


Physically implemented


How Teleportation Works (Roughly)

1. Two parties: Alice and Bob, sharing an EPR pair.
2. Alice teleports a particle to Bob in that she measures a certain state and informs Bob about the result via a classical channel.
The **categorical** expression of Quantum Teleportation:

(B. Coecke, D. Pavlović, 2006)
Superdense Coding

Introduced in


How Superdense Coding Works (Roughly)

1. Two parties: Alice and Bob, sharing an EPR pair.
2. Alice encodes two classical bits into one q-bit and sends it to Bob via a quantum channel. Bob retrieves the message in that he measures a certain state.
Superdense Coding in Picture Calculus:

(B. Coecke, D. Pavlović, 2006)
Support from the Fathers of QM

...I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space anymore.

John von Neumann in a letter to George David Birkhoff, 13 November 1935

And Remember

In mathematics you don't understand things. You just get used to them.

John von Neumann (1903–1957)
**Primitive Notions**

A (Labelled) Box: \[ f \]  
A (Labelled) Wire: \[ V \]

**Basic Axioms**

The Void Wire: \[ I \]  
The Involution: \[ V^* \]

The void wire can be **omitted** from any picture.

**Operators**

These are made from wires and boxes, e.g.,

\[
\begin{align*}
V^* & \quad f & V & \quad l & \quad f \\
V & \quad W & & \quad ket & \quad V & \quad f \\
& & & \quad V & \quad V & \quad f \\
& & & \quad l & \quad f & \quad scalar
\end{align*}
\]
In Classical Model

\( V = \) finitely dimensional complex Hilbert space
\( V^* = \) space conjugate to \( V \)
\( I = \) the 1-dimensional space (complex numbers)

\[ f : \mathbb{C} \to V, \text{ i.e., } |f\rangle \]

\[ f : V \to W \]

\[ f : W \to V^* \]
In Classical Model

linear map from $V$ to $\mathbb{C}$, i.e., $\langle f \rangle$

linear map from $\mathbb{C}$ to $\mathbb{C}$, i.e., a scalar
Sequential Composition

Glue together the corresponding wires: this composition is associative and has units:

Thus, $1_V$ can be replaced by wire $V$ in any picture.
Parallel Composition

\[ W \xrightarrow{f} V \quad \text{and} \quad W' \xrightarrow{g} V' \]

Lemma

\[ 1_I \cdot f = f = f \cdot 1_I \]

\[ f \cdot g = g \cdot f \]
Adjoints Operators

For every operator $f$ there is a unique adjoint $f^\dagger$ obtained just by
symmetry along the centre of the box.

For example:

\[\begin{array}{c}
V^* \\
\uparrow \\
\downarrow \\
\downarrow \\
W \\
\uparrow \\
f \\
\uparrow \\
\downarrow \\
\downarrow \\
V \\
\uparrow \\
\uparrow \\
V^* \\
\end{array}\]

its adjoint is

\[\begin{array}{c}
W \\
\uparrow \\
\downarrow \\
\downarrow \\
V \\
\uparrow \\
\uparrow \\
V^* \\
\end{array}\]
In Classical Model

adjoint = transpose of the conjugate

Observe:

for kets $\alpha$ and $\beta$, the scalar $V$ is the inner product $\langle \beta | \alpha \rangle$. 
Bell States and Yanking

For every $V$ there is a Bell state

such that Yanking Axioms hold:
This is Already Quite a Powerful Beast

1. One can prove basic facts about transposes, adjoints, unitary, self-adjoint and positive operators.
2. Traces can be defined.
3. One can prove the Hilbert-Schmidt Correspondence.
4. Spectral Decomposition Theorem and Born’s Rule can be derived. (Requires biproducts.)
5. . . . and more.

But there is a serious drawback: linear algebra sneaks in!
The Calculus at Work, No 1: The Hilbert-Schmidt Correspondence

The map

\[ W \xrightarrow{f} V \mapsto \eta \downarrow V \]

is a bijection.

In Classical Model

There is a bijection

\[ \text{Lin}(V, W) \cong V^* \otimes W \]
The Calculus at Work, No 2: The No-Cloning Theorem

Suppose

\[
\begin{align*}
V & V \\
\alpha & \varphi \\
U & = \\
V & V \\
\alpha & \alpha \\
& = \\
V & V \\
\beta & \varphi \\
& = \\
V & V \\
\beta & \beta \\
\end{align*}
\]

holds for states \(|\alpha\rangle\), \(|\beta\rangle\) and \(|\varphi\rangle\). Then we have the equality

\[
\begin{align*}
V & V \\
\alpha & \alpha \\
= \\
V & \\
\alpha & \alpha \\
\Rightarrow \\
\beta & \\
& \Rightarrow \\
\end{align*}
\]
Proof of The No-Cloning Theorem

\[
\begin{align*}
\beta\quad &\quad \beta^* \\
V &\quad V \\
\alpha &\quad \alpha \\
\downarrow &\quad \downarrow \\
\alpha &\quad \varphi \\
V &\quad V \\
\beta^* &\quad \varphi^* \\
\uparrow &\quad \uparrow \\
\beta &\quad \varphi \\
\downarrow &\quad \downarrow \\
\beta^* &\quad \varphi^* \\
V &\quad V \\
\alpha &\quad \varphi \\
\downarrow &\quad \downarrow \\
1 &\quad =
\end{align*}
\]
The Goal

To get rid of linear algebra once for all. This is done by distinguishing classical from quantum.

What does Distinguish Classical from Quantum?

Classical data can be copied and deleted:

In quantum world, Bell States prohibit copying (entanglement).
Entanglement Prohibits Copying

\[ |1\rangle \rightarrow |0\rangle + |1\rangle \]

\[ |1\rangle \otimes |1\rangle \rightarrow (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \neq (|0\rangle \otimes |0\rangle) + (|0\rangle \otimes |0\rangle) \]
Axioms For Copying and Deleting

\[ \begin{align*}
X & \xrightarrow{\delta} X = X \\
X & \xrightarrow{\delta} X = X \\
X & \xrightarrow{\delta} X = X \\
X & \xrightarrow{\delta} X = X
\end{align*} \]
This Allows Us to Avoid Linear Algebra Altogether


References

1. B. Coecke: *Kindergarten Quantum Mechanics*,

2. S. Abramsky and B. Coecke: *Categorical Semantics of Quantum Protocols*,
