Random processes

Definition of random process

Definition

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $T \subset \mathbb{R}$. The system of real random variables $\{X_t, t \in T\}$ defined on $(\Omega, \mathcal{A}, P)$ is called the random (or stochastic) process.

Definition

If $T = \mathbb{Z}$ or $T = \mathbb{N}$, we talk about the random process with discrete time. If $T = [a, b]$, where $-\infty \leq a < b \leq \infty$, we talk about the random process with continuous time.

Definition

Double $(S, \mathcal{E})$ is called the state space, if S is a set of values of the random variable X_t and $\mathcal{E}$ is $\sigma-$algebra on the S.

Definition

If the random variables X_t take only discrete values, we talk about the random process with discrete states. If the random variables X_t take continuous values, we talk about the random process with continuous states.

Random process $\{X_t, t \in T\}$ can be considered as a function of two variables ω and t. For fixed t, this function is a random variable, for fixed ω it is a function of one real variable t.

Definition

Consider a fixed $\omega \in \Omega$. Then the function $t \to X_t$ is called the trajectory of the process $\{X_t, t \in T\}$.

Definition

The proces is called continuous, if all its trajectories are continuous.

Weak stationarity of the process

Definition

Let $\{X_t, t \in T\}$ be a random process such that for all $t \in T$ there exists $\mathbb{E}X_t$. Then the function $\mu_t = \mathbb{E}X_t$ defined on T is called expected value of the process $\{X_t\}$. If $\mathbb{E}|X_t|^2 < \infty$ for all $t \in T$, then the function defined on $T \times T$ as $R(s,t) = \mathbb{E}(X_s - \mu_s)(X_t - \mu_t)$ is called autocovariance function of the process $\{X_t\}$. The value $R(t,t)$ is called the variance of the process $\{X_t\}$ in time t.

Definition

The random process $\{X_t, t \in T\}$ is called to be weakly stationary if $R(s,t)$ is the function of the difference $s - t$, i.e.

$$R(s,t) = \tilde{R}(s-t).$$

Corollary:

$$R(s,t) = R(s+h, t+h)$$

for all $h \in \mathbb{R}$ such that $s + h \in T$ and $t + h \in T$.

Strict stationarity of the process

Denote

$$F_{t_1,\dots,t_n}(x_1,\dots,x_n) = P(X_{t_1} \le x_1,\dots,X_{t_n} \le x_n).$$

Definition

The random process $\{X_t, t \in T\}$ is called to be strictly stationary if for arbitrary $n \in \mathbb{N}$, arbitrary real $x_1,\dots,x_n$ and arbitrary real $t_1,\dots,t_n$ and h such that $t_k \in T$, $t_k + h \in T$, $1 \le k \le n$, it holds that

$$F_{t_1,\dots,t_n}(x_1,\dots,x_n) = F_{t_1+h,\dots,t_n+h}(x_1,\dots,x_n). \tag{1}$$

Remark

For processes with discrete states, the relation (1) is equivalent to the relation

$$P(X_{t_1} = x_1,\dots,X_{t_n} = x_n) = P(X_{t_1+h} = x_1,\dots,X_{t_n+h} = x_n).$$

Consider

- a probability space $(\Omega, \mathcal{A}, P)$,
- a sequence of random variables $\{X_n, n \in \mathbb{N}\}$ defined on that space,
- state space $(S, \mathcal{E})$, where the set S is finite or countably infinite, so without loss of generality suppose $S = \{0, 1, \dots, N\}$, resp. $S = \{0, 1, \dots\}$.

Definition

The sequence of the random variables $\{X_n, n \in \mathbb{N}\}$ is called Markov chain with discrete time, if

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = P(X_{n+1} = j | X_n = i)$$

for all $n = 0, 1, \dots$ and all $i, j, i_{n-1}, \dots, i_0 \in S$ such that $P(X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) > 0$.

Example: random walk

Let $Y_1, Y_2, \ldots$ be independent, identically distributed random variables taking values ± 1 with probabilities $1/2$.

Define

$$X_0 = 0$$
$$X_n = \sum_{i=1}^{n} Y_i.$$

Then the sequence (process, chains) $\{X_n, n \in \mathbb{N}\}$ is called *the random walk*.

Definition

Conditional probabilities

1. $P(X_{n+1} = j | X_n = i) = p_{ij}(n, n+1)$ are called the transit probabilities from the state i in time n to the state j in time $n+1$ or the transit probabilities of the first order;
2. $P(X_{n+m} = j | X_n = i) = p_{ij}(n, n+m)$ are called the transit probabilities from the state i in time n to the state j in time $n+m$ or the transit probabilities of the m-th order.

Definition

If the transit probabilities $p_{ij}(n, n+m)$ do not depend on the times n and $n+m$, but only on the difference m, the Markov chains is called to be homogeneous.

Matrix of the transit probabilities

- Consider a homogeneous chain and denote $p_{ij} := p_{ij}(n, n+1)$.
- These elements can be ordered into a matrix $\mathbf{P} = \{p_{ij}, i, j \in S\}$, where it holds that

$$p_{ij} \geq 0, \forall i, j \in S \quad \text{and} \quad \sum_{j \in S} p_{ij} = 1, \forall i \in S.$$

Definition

The matrix $\mathbf{P} = \{p_{ij}, i, j \in S\}$ is called the matrix of the transit probabilities.

Denote

$$p_i = P(X_0 = i), \quad \forall i \in S.$$

Obviously, it holds that

$$p_i \geq 0, \forall i \in S \quad \text{and} \quad \sum_{i \in S} p_i = 1.$$

Definition

The vector $\mathbf{p} = \{p_i, i \in S\}$ is called initial distribution of the Markov chain.

It follows from the Chain rule that

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = p_{i_0} p_{i_0 i_1} \ldots p_{i_{n-1} i_n}.$$

Denote $p_{ij}^{(1)} = p_{ij}$ and for positive integers $n \geq 1$, define subsequently

$$p_{ij}^{(n+1)} = \sum_{k \in S} p_{ik}^{(n)} p_{kj}. \tag{2}$$

It can be shown that $p_{ij}^{(n)} \leq 1$ and moreover for the matrix of the transit probabilities it holds that

$$\mathbf{P}^{(2)} = \mathbf{P} \cdot \mathbf{P} = \mathbf{P}^2 \text{ and generally } \mathbf{P}^{(n+1)} = \mathbf{P}^{(n)} \cdot \mathbf{P} = \mathbf{P} \cdot \mathbf{P}^{(n)} = \mathbf{P}^{n+1}.$$

Theorem

Let $\{X_n, n \in \mathbb{N}\}$ be a homogeneous Markov chain with matrix of the transit probabilities $\mathbf{P}$. Then for the transit probabilities of the $n-$th order, it holds that

$$P(X_{m+n} = j | X_m = i) = p_{ij}^{(n)}, \quad \forall i, j \in S$$

for all positive integer m and n and for $P(X_m = i) > 0$.

The relation (2) can be generalised to

$$p_{ij}^{(m+n)} = \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)},$$

i.e.

$$\mathbf{P}^{(m+n)} = \mathbf{P}^{(m)} \cdot \mathbf{P}^{(n)}.$$

This generalisation is called Chapman-Kolmogorov equality.

Notation

- If the chain $\{X_n, n \in \mathbb{N}\}$ starts from the state j, i.e. $P(X_0 = j) = 1$, then we denote
$$P(.|X_0 = j) = P_j(.).$$
- Define the random variable
$$\tau_j(1) = \inf\{n > 0 : X_n = j\}$$
as the time of the first return of the chain to the state j.
- Denote $\mu_j = \mathbb{E}[\tau_j(1)|X_0 = j]$ the expected value of the time of the first return of the chain to the state j.
- Denote d_j the highest divisor of the times $n \geq 1$ such that $p_{jj}^{(n)} > 0$.

Definition

The state j of the Markov chain is called recurrent if

$$P_j(\tau_j(1) < \infty) = 1.$$

The state j of the Markov chain is called transient if

$$P_j(\tau_j(1) = \infty) > 0.$$

Definition

The recurrent state j of the Markov chain is called nonnull recurrent, if $\mu_j < \infty$ and null recurrent, if $\mu_j = \infty$.

Definition

If $d_j > 1$, the state j of the Markov chain is called periodic with period d_j, if $d_j = 1$, the state j of the Markov chain is called aperiodic.

Theorem

a) *Let j be the transient state. Then* $\lim_{n\to\infty} p_{ij}^{(n)} = 0, \forall i \in S$.

b) *Let j be the null recurrent state. Then* $\lim_{n\to\infty} p_{ij}^{(n)} = 0, \forall i \in S$.

c) *Let j be the nonnull recurrent and aperiodic the state. Then* $\lim_{n\to\infty} p_{jj}^{(n)} = \frac{1}{\mu_j}$.

d) *Let j be the nonnull recurrent state with the period d_j. Then* $\lim_{n\to\infty} p_{jj}^{(nd_j)} = \frac{d_j}{\mu_j}$.

Theorem

The recurrent state j is null if and only if $\lim_{n\to\infty} p_{jj}^{(n)} = 0$.

Definition

The state j is reachable (or accesible) from the state i, if there exists $n \in \mathbb{N}$ such that $p_{ij}^{(n)} > 0$.

Definition

A set C of the states is called to be closed, if none of the state lying outside the set C is reachable from any state included inside the set C.

Theorem

The set C of the states is closed if and only if $p_{ij} = 0$ for all $i \in C, j \notin C$.

Definition

Markov chain is called to be irreducible, if each of its states is reachable from each of the remaining states. Otherwise, it is called to be reducible.

Definition

If one-element set of the states $\{j\}$ is closed, i.e. $p_{jj} = 1$, then the state j is called the absorption state.

Definition

Let $\{X_n, n \in \mathbb{N}\}$ be a homogeneous chain with the set of the states S and matrix of the transit probabilities $\mathbf{P}$. Let $\pi = \{\pi_j, j \in S\}$ be a probability distribution on S, i.e. $\pi_j \geq 0, j \in S, \sum_{j\in S} \pi_j = 1$. Then π is called the stationary distribution of the chain, if it holds

$$\pi^T = \pi^T \mathbf{P},$$

i.e.

$$\pi_j = \sum_{k\in S} \pi_k p_{kj}, j \in S.$$

Theorem

Let the initial distribution of a homogeneous Markov chains be stationary. Then this chain is stationary and moreover for all $n \in \mathbb{N}$, it holds that

$$p_j(n) = P(X_n = j) = \pi_j, \quad j \in S,$$

where π_j are the initial probabilities.

Theorem

For an irreducible Markov chain, it holds that:

1. *If all its states are transient or null recurrent, the stationary distribution does not exist.*
2. *If all its states are nonnull recurrent, the stationary distribution exists and it is unique.*
    1. *If all its states are aperiodic, then for all $i, j \in S$, it holds that*

    $$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)} > 0 \quad \textit{and} \quad \pi_j = \lim_{n\to\infty} p_j(n) > 0.$$

    2. *If all its states are periodic, then for all $i, j \in S$, it holds that*

    $$\pi_j = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)} > 0 \quad \textit{and} \quad \pi_j = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} p_j(k) > 0.$$

3. *In irreducible chain with finite number of states, there always exists the stationary distribution.*